Bifurcation Analysis of DDEs

Simplest local bifurcations.
Critical normal forms for codim 1 bifurcations of equilibria.

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Contents

Literature

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1. Simplest critical equilibria and their computation

Consider a **DDE** with m delays for $x(t) \in \mathbb{R}^n$ and parameter $\alpha \in \mathbb{R}$:

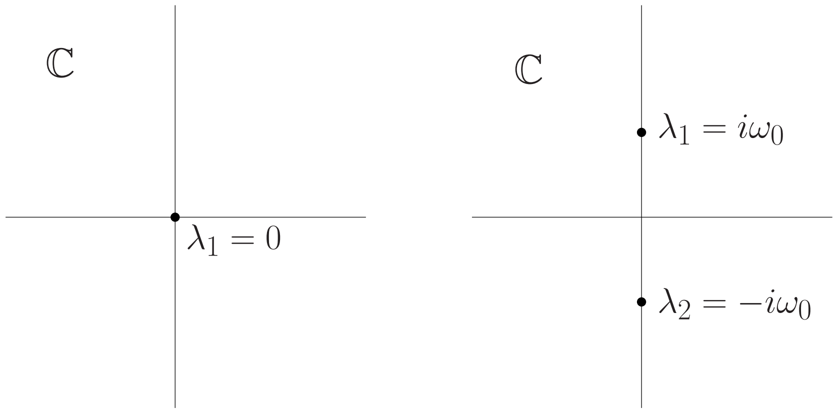
$$\dot{x}(t) = f(x(t), x(t-\tau_1), x(t-\tau_2), \ldots, x(t-\tau_m), \alpha),$$

where $f : \mathbb{R}^{(m+1)n} \times \mathbb{R} \to \mathbb{R}^n$ is smooth and $0 = \tau_0 < \tau_1 < \cdots < \tau_m = h$. Assume that $f(0) := f(0,0,0,\ldots,0,0) = 0$, i.e. $x = 0$ is an **equilibrium** at $\alpha = 0$.

- Let $\lambda_j \in \mathbb{C}$ be roots of the **characteristic equation** $\det \Delta(\lambda) = 0$,

$$\Delta(\lambda) = \lambda I_n - \sum_{j=0}^{m} A_j e^{-\lambda \tau_j}, \quad A_j = D_j f(0,0), j = 0, 1, \ldots, m.$$

- Codim 1 **critical cases** for stability: $\Re(\lambda) = 0$



Defining systems

Let
$$\Delta(\lambda, u, \alpha) := \lambda I_n - \sum_{j=0}^{m} \tilde{A}_j(u, \alpha) e^{-\lambda \tau_j},$$

where $\tilde{A}_j(u, \alpha) := D_j f(u, u, u, \ldots, u, \alpha),\ \tilde{A}_j(0, 0) = A_j,\ j = 0, 1, \ldots, m.$

- **Fold**: $\lambda_1 = 0$
$$\begin{cases} f(u, u, u, \ldots, u, \alpha) &= 0, \\ \Delta(0, u, \alpha) q &= 0, \\ cq &= 1, \end{cases}$$
where $(q, u, \alpha) \in \mathbb{R}^{2n+1}, c \in \mathbb{R}^{n*}$.

- **Andronov-Hopf**: $\lambda_{1,2} = \pm i\omega_0,\ \omega_0 > 0$
$$\begin{cases} f(u, u, u, \ldots, u, \alpha) &= 0, \\ \Delta(i\omega_0, u, \alpha) q &= 0, \\ cq &= 1, \end{cases}$$
where $(q, u, \alpha, \omega_0) \in \mathbb{C}^n \times \mathbb{R}^n \times \mathbb{R}^2, c \in \mathbb{C}^{n*}$.

2. Center manifold reduction

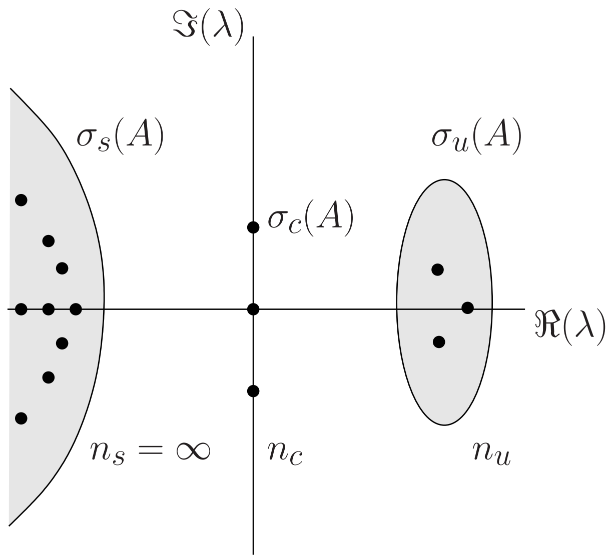
- Let $T(t)$ be the **semigroup** on $X = C([-h,0],\mathbb{R}^n)$ defined by

$$\dot{y}(t) = A_0 y(t) + \sum_{j=1}^{m} A_j y(t-\tau_j),$$

and A its **infinitesimal generator**: $(A\phi)(\theta) = \dot{\phi}(\theta)$ for $\phi \in D(A)$,

$$D(A) = \{\phi \in X : \dot{\phi} \in X \quad \text{and} \quad \dot{\phi}(0) = \sum_{j=0}^{m} A_j \phi(-\tau_j)\}.$$

- Suppose that A has n_c **critical eigenvalues**/characteristic roots:

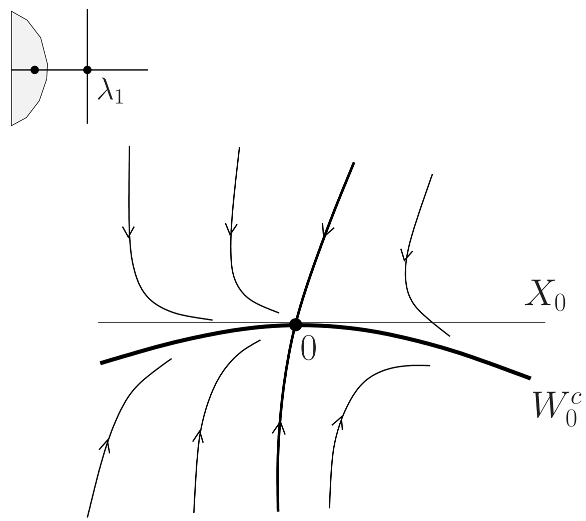


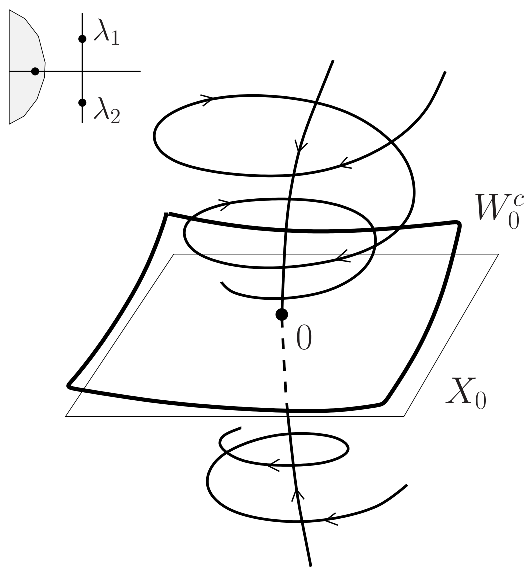
Center Manifold Theorem

- Let $S_\alpha(t) : X \to X$ be the semigroup generated by a smooth DDE

$$\dot{x}(t) = f(x(t), x(t-\tau_1), x(t-\tau_2), \ldots, x(t-\tau_m), \alpha).$$

- Suppose, at $\alpha = 0$ the equilibrium $x = 0$ has critical eigenvalues. Denote by X_0 the (generalized) critical eigenspace of A with $\dim X_0 = n_c < \infty$.
- For each sufficiently small $|\alpha|$, there exists a smooth n_c-dimensional manifold W^c_α (called **center manifold**) that is locally invariant and normally hyperbolic for $S_\alpha(t)$. Moreover, W^c_0 is tangent at $x = 0$ to X_0.





3. Local bifurcations in one-parameter DDEs

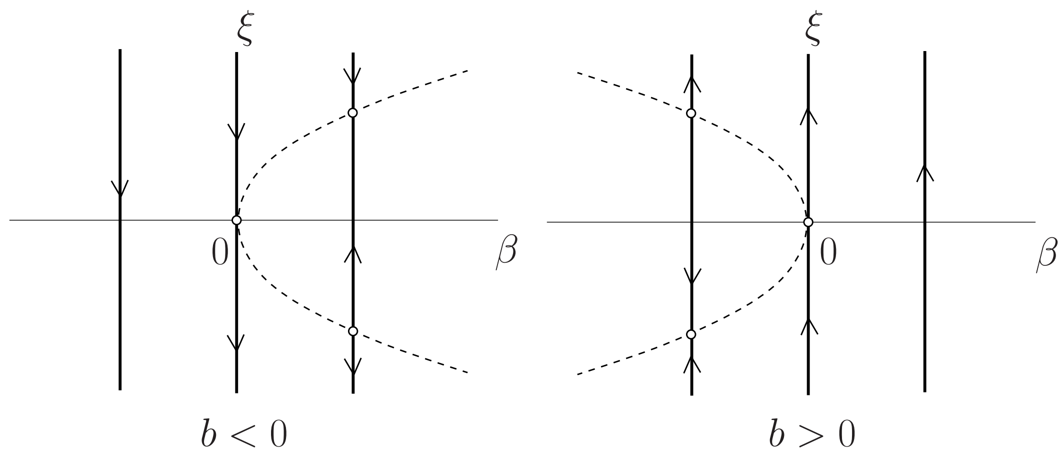
The restriction of $S_\alpha(t)$ to W^c_α is locally generated by a smooth ODE

$$\dot{\xi} = g(\xi, \alpha),\ \ \xi \in \mathbb{R}^{n_c}, \alpha \in \mathbb{R}.$$

Fold:

- ODE on $W^c_0 : \dot{\xi} = b_0 \xi^2 + \ldots,\ \ \xi \in \mathbb{R}$
- Smooth normal form on W^c_α when $b_0 \neq 0$:

$$\dot{\xi} = \beta(\alpha) + b(\alpha)\xi^2 + \ldots, \quad \beta(0) = 0, b(0) = b_0.$$

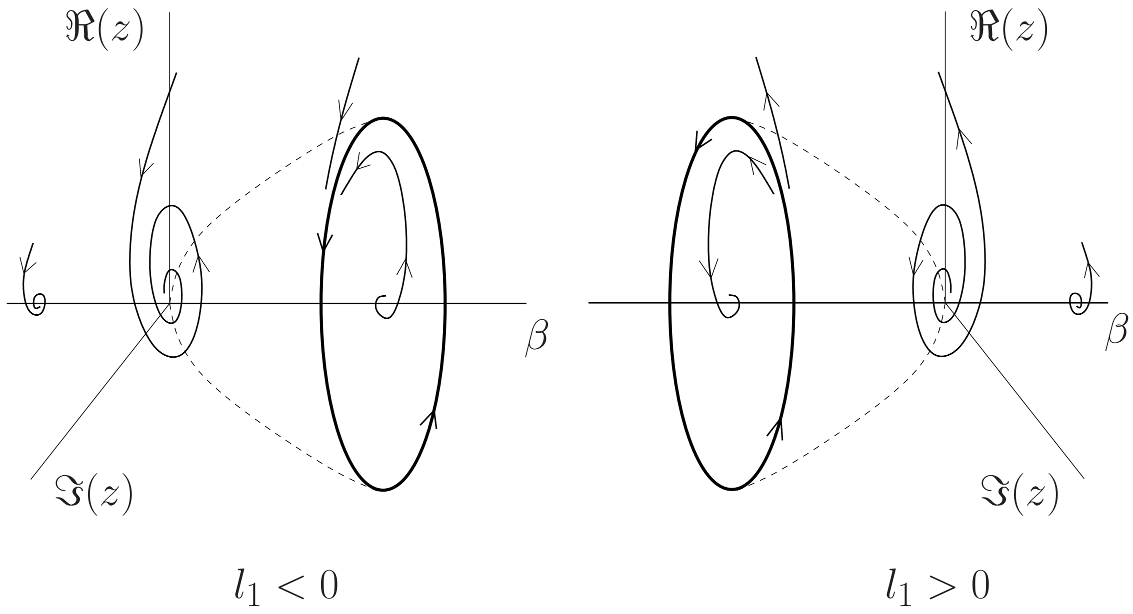


Equilibria: $\xi_{1,2} \approx \pm\sqrt{-\dfrac{\beta}{b}}$

Andronov-Hopf:

- Normalized ODE on W_0^c: $\dot{z} = i\omega_0 z + c_1 z|z|^2 + \dots,\ \omega_0 > 0, z \in \mathbb{C}$.
- Smooth normal form on W_α^c when $l_1 := \frac{1}{\omega_0}\Re(c_1) \neq 0$:

$$\dot{z} = (\beta(\alpha) + i\omega(\alpha))z + c(\alpha)z|z|^2 + \dots,\quad \beta(0) = 0, \omega(0) = \omega_0, c(0) = c_1.$$



Limit cycle: $\begin{cases} \dot{\rho} = \rho(\beta + \Re(c)\rho^2) + \dots, \\ \dot{\varphi} = \omega + \Im(c)\rho^2 + \dots, \end{cases} \Rightarrow \rho_0 \approx \sqrt{-\frac{\beta}{\Re(c)}}$

4. The fold critical normal form coefficient

- Let $G(u,\alpha) := -f(u,u,u,\ldots,u,\alpha)$, so that
$$J = D_u G(0,0) = -A_0 - \sum_{j=1}^{m} A_j = \Delta(0),$$
where $A_j = D_j f(0), j = 0,1,\ldots,m$.

- Let $q \in \mathbb{R}^n, p \in \mathbb{R}^{n*}$ be such that $Jq = 0$ and $pJ = 0$ with $p\Delta'(0)q \neq 0$.

- There is a coordinate $\xi \in \mathbb{R}$ on W_0^c such that the **normal form coefficient**
$$b_0 = \frac{1}{2} p\, D_u^2 G(0,0)(q,q).$$
This follows from the approximation of the curve $G(u,\alpha) = 0$ near $(u,\alpha) = (0,0) \in \mathbb{R}^{n+1}$, which is the **finite-dimensional** problem.

5. Sun-star calculus

- Consider a **nonlinear DDE**

$$\dot{x}(t) = Lx_t + F(x_t), \quad x_t \in X = C([-h, 0], \mathbb{R}^n),$$

where $F : X \to \mathbb{R}^n$ is smooth and contains only nonlinear terms, while

$$L\phi = \int_0^h d\zeta(\theta)\phi(-\theta), \quad \zeta \in \mathsf{NBV}([0, h], \mathbb{R}^{n\times n}).$$

Here $\mathsf{NBV}([0, h], \mathbb{R}^{n\times n})$ is the space of normalized bounded-variation matrix-valued functions.

- The **linearized DDE** $\dot{y} = Lx_t$ defines the strongly continuous semigroup $T(t) : X \to X$ with the infinitesimal generator A. Its characteristic matrix can now be written as

$$\Delta(\lambda) = \lambda I_n - \int_0^h e^{-\lambda\theta} d\zeta(\theta)$$

Duality:

- Let $T^*(t) : X^* \to X^*$ be the **adjoint semigroup**, i.e. for $t \geq 0$

$$\langle T^*(t)\phi^*, \phi\rangle = \langle \phi^*, T(t)\phi\rangle, \quad \phi^* \in X^*, \phi \in X.$$

  Denote by $X^\odot$ the maximal subspace of X^* on which $T^*(t)$ is strongly continuous, and define

$$T^\odot(t) = T^*(t)|_{X^\odot}$$

  Denote its infinitesimal generator by $A^\odot$.

- Let $A^{\odot\star}$ be the generator of the adjoint semigroup $T^{\odot\star}(t) : X^{\odot\star} \to X^{\odot\star}$. Define $X^{\odot\odot}$ as the maximal subspace of $X^{\odot\star}$ on which $T^{\odot\star}(t)$ is strongly continuous.

- Introduce the **embedding** $j : X \to X^{\odot\star}$ by

$$\langle jx, x^\odot\rangle := \langle x^\odot, x\rangle, \quad \forall x \in X, \forall x^\odot \in X^\odot.$$

  In general, $j(X) \subset X^{\odot\odot}$. However, the space $X = C([-h, 0], \mathbb{R}^n)$ is **sun-reflexive**: $j(X) = X^{\odot\odot}$.

Concrete representations:

- **Spaces:**

space	representation	duality pairing
X X^*	$\phi \in C([-h,0],\mathbb{R}^n)$ $\eta \in \mathrm{NBV}([0,h],\mathbb{R}^{n*})$	$\langle \eta, \phi \rangle = \int_0^h d\eta(\theta)\phi(-\theta)$
X $X^\odot$	$\phi \in C([-h,0],\mathbb{R}^n)$ $(c,g) \in \mathbb{R}^{n*} \times L^1([0,h],\mathbb{R}^{n*})$	$\langle (c,g), \phi \rangle = c\phi(0) + \int_0^h g(\theta)\phi(-\theta)\, d\theta$
$X^\odot$ $X^{\odot\star}$	$(c,g) \in \mathbb{R}^{n*} \times L^1([0,h],\mathbb{R}^{n*})$ $(\alpha,\psi) \in \mathbb{R}^n \times L^\infty([-h,0],\mathbb{R}^n)$	$\langle (\alpha,\psi), (c,g) \rangle = c\alpha + \int_0^h g(\theta)\psi(-\theta)\, d\theta$

- **Embedding:**

$$j\phi = (\phi(0), \phi) \in X^{\odot\star} \quad , \phi \in X.$$

- **Nonlinearity** $R : X \to X^{\odot\star}$ is defined by

$$R(\phi) := \sum_{i=1}^{n} F_i(\phi) r_i^{\odot\star}, \quad \phi \in X,$$

where

$$r_i^{\odot\star} := (e_i, 0) \in X^{\odot\star}, \quad i = 1, \ldots, n,$$

and e_i is the i-th standard basis vector in $\mathbb{R}^n$.

6. The first Lyapunov coefficient for Andronov-Hopf bifurcation

- The solution $u(t) := x_t \in W_0^c \subset X$ satisfies a well-defined **ODE** in $X^{\odot\star}$:

$$\frac{d}{dt} ju(t) = A^{\odot\star} ju(t) + R(u(t)),$$

where $R : X \to X^{\odot\star}$ can be expanded as

$$R(u) = \frac{1}{2}B(u,u) + \frac{1}{6}C(u,u,u) + O(\|u\|^4).$$

- The parametrization of W_0^c: $u = \mathcal{H}(z,\bar{z})$ with

$$\mathcal{H}(z,\bar{z}) = z\phi + \bar{z}\bar{\phi} + \sum_{2\leq j+k\leq 3} \frac{1}{j!k!} h_{jk} z^j \bar{z}^k + O(|z|^4), \quad z \in \mathbb{C},$$

where $A\phi = i\omega_0\phi$, $A^*\phi^{\odot} = i\omega_0\phi^{\odot}$, $\langle \phi^{\odot}, \phi \rangle = 1$.

- Poincaré normal form on $W_0^c : \dot{z} = i\omega_0 z + c_1 z|z|^2 + O(|z|^4)$, $z \in \mathbb{C}$.

- **Homological equation**

$$j\left(D_z\mathcal{H}(z,\overline{z})\dot{z} + D_{\overline{z}}\mathcal{H}(z,\overline{z})\dot{\overline{z}}\right) = A^{\odot\star} j\mathcal{H}(z,\overline{z}) + R(\mathcal{H}(z,\overline{z}))$$

- **Qudratic terms**

$$\begin{aligned} z^2 &: \ -A^{\odot\star} j h_{20} \ = \ B(\phi,\overline{\phi}), \\ z\overline{z} &: \ (2i\omega_0 - A^{\odot\star}) j h_{11} \ = \ B(\phi,\phi), \end{aligned}$$

which are uniquely solvable and define h_{20} and h_{11}.

- **Resonance cubic term**

$$z^2\overline{z} \ : \ (i\omega_0 I - A^{\odot\star}) j h_{21} \ = \ C(\phi,\phi,\overline{\phi}) + B(\overline{\phi}, h_{20}) + 2B(\phi, h_{11}) - 2c_1 j\phi.$$

This system is **singular**. Pairing with $\phi^{\odot}$ gives

$$c_1 = \frac{1}{2}\langle \phi^{\odot}, C(\phi,\phi,\overline{\phi}) + B(\overline{\phi}, h_{20}) + 2B(\phi, h_{11})\rangle$$

- The **first Lyapunov coefficient** $l_1 = \dfrac{1}{\omega_0}\Re(c_1)$.

Computational formulas:

- Eigenfunctions

$$\begin{aligned} \phi(\theta) &= e^{i\omega_0\theta} q, \\ \phi^{\odot} &= \left(p, p\int_\theta^h e^{i\omega_0(\theta-\tau)} d\zeta(\tau)\right) \end{aligned}$$

where $q \in \mathbb{C}^n, p \in \mathbb{C}^{n*}$ satisfy

$$\Delta(i\omega_0)q = 0,\ \ p\Delta(i\omega_0) = 0,\ \ p\Delta'(i\omega_0)q = 1.$$

- Quadratic coefficients

$$\begin{aligned} h_{20} &= e^{2i\omega_0\theta}\Delta(2i\omega_0)^{-1}D^2F(0)(\phi,\phi) \\ h_{11} &= \Delta(0)^{-1}D^2F(0)(\phi,\overline{\phi}) \end{aligned}$$

- The normal form coefficient

$$c_1 = \frac{1}{2}p\Big[D^2F(0)(\overline{\phi}, e^{2i\omega_0\theta}\Delta(2i\omega_0)^{-1}D^2F(0)(\phi,\phi)) + 2D^2F(0)(\phi, \Delta(0)^{-1}D^2F(0)(\phi,\overline{\phi})) + D^3F(0)(\phi,\phi,\overline{\phi})\Big]$$

(implemented in **DDE-BIFTOOL** to compute the first Lyapunov coefficient l_1).

Computation of derivatives: DDE at the critical parameter values:

$$\dot{x}(t) = f(x(t), x(t-\tau_1), x(t-\tau_2), \ldots, x(t-\tau_m))$$

- Recall that $f : \mathbb{R}^{n(m+1)} \to \mathbb{R}^n$ with

  $X := (x^0, x^1, x^2, \ldots, x^m) \mapsto f(x^0, x^1, x^2, \ldots, x^m),\ \ x^j \in \mathbb{R}^n, j = 0, 1, 2, \ldots, m.$

- The (multi-)linear forms:

  For $Q, P, R \in \mathbb{R}^{n(m+1)}$ with components q_k^j, p_k^j, r_k^j define

$$\begin{aligned} D^2 f^0(Q, P) &:= \sum_{k_1, k_2 = 1}^{n} \sum_{j_1, j_2 = 0}^{m} \frac{\partial^2 f(0)}{\partial x_{k_1}^{j_1} \partial x_{k_2}^{j_2}} q_{k_1}^{j_1} p_{k_2}^{j_2} \\ D^3 f^0(Q, P, R) &:= \sum_{k_1, k_2, k_3 = 1}^{n} \sum_{j_1, j_2, j_3 = 0}^{m} \frac{\partial^3 f(0)}{\partial x_{k_1}^{j_1} \partial x_{k_2}^{j_2} \partial x_{k_3}^{j_3}} q_{k_1}^{j_1} p_{k_2}^{j_2} r_{k_3}^{j_3} \end{aligned}$$

Computation of derivatives:

$$F : C([-h,0],\mathbb{R}^n) \to \mathbb{R}^n,\ \ F(\phi) = f(\phi(0),\phi(-\tau_1),\phi(-\tau_2),\ldots,\phi(-\tau_m))$$

- 2nd Differentials:

$$\begin{aligned} D^2F(0)(\phi,\phi) &= D^2f^0(\Phi,\Phi), \\ D^2F(0)(\phi,\overline{\phi}) &= D^2f^0(\Phi,\overline{\Phi}), \\ D^2F(0)(\overline{\phi},h_{20}) &= D^2f^0\left(\bar{\Phi},H_{20}\right), \\ D^2F(0)(\phi,h_{11}) &= D^2f^0\left(\Phi,H_{11}\right), \end{aligned}$$

where

$$\begin{aligned} \Phi &= \left(\phi(0),\phi(-\tau_1),\ldots,\phi(-\tau_m)\right), \\ H_{20} &= \left(h_{20}(0),h_{20}(-\tau_1),\ldots,h_{20}(-\tau_m)\right), \\ H_{11} &= \left(h_{11}(0),h_{11}(-\tau_1),\ldots,h_{11}(-\tau_m)\right). \end{aligned}$$

- 3rd Differential:

$$D^3F(0)(\phi,\phi,\overline{\phi}) = D^3f^0(\Phi,\Phi,\overline{\Phi}).$$