Numerical Continuation and Normal Form Analysis of Limit Cycle Bifurcations without Computing Poincaré Maps

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joint work with *W. Govaerts*, *A. Dhooge*(Gent), and *E. Doedel* (Montreal)

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1. References

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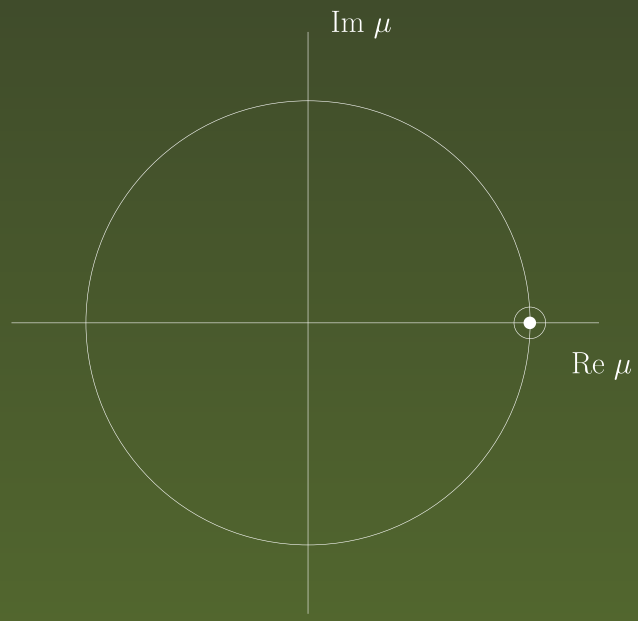
2. Limit cycles of ODEs and their local bifurcations

$$\frac{dx}{dt} = f(x,\alpha), \;\; x \in \mathbb{R}^n, \alpha \in \mathbb{R}^m.$$

A limit cycle C_0 corresponds to a periodic solution $x_0(t+T_0) = x_0(t)$ and has Floquet multipliers $\{\mu_1, \mu_2, \ldots, \mu_{n-1}, \mu_n = 1\} = \sigma(M(T_0))$, where

$$\dot{M}(t) - f_x(x_0(t), \alpha_0) M(t) = 0, \;\; M(0) = I_n.$$



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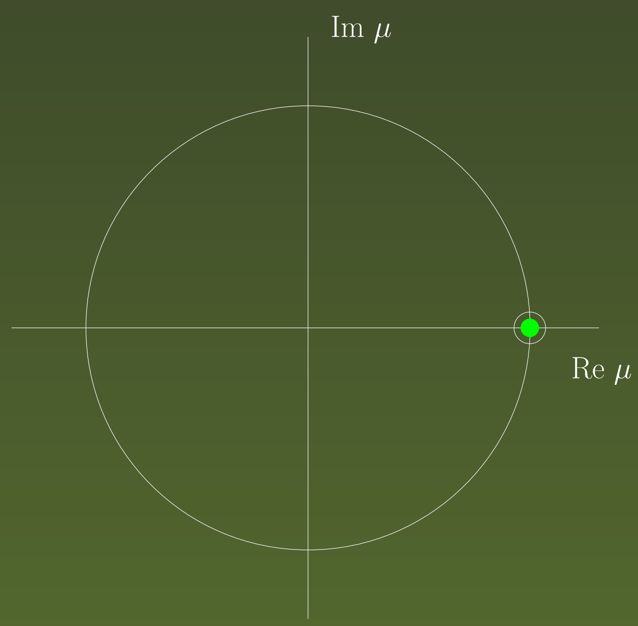
- Fold (LPC): $\mu_1 = 1$;



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$$\frac{dx}{dt} = f(x, \alpha), \quad x \in \mathbb{R}^n, \alpha \in \mathbb{R}^m.$$

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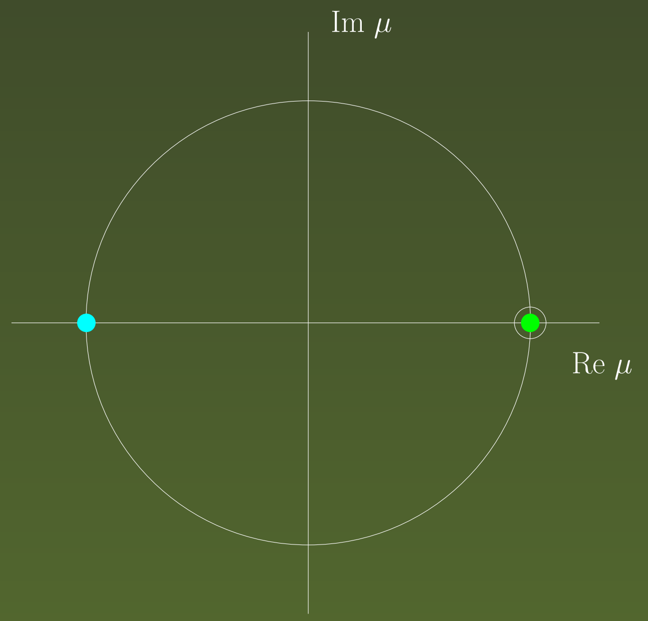
- Fold (LPC): $\mu_1 = 1$;
- Flip (PD): $\mu_1 = -1$;



2. Limit cycles of ODEs and their local bifurcations

$$\frac{dx}{dt} = f(x,\alpha), \quad x \in \mathbb{R}^n, \alpha \in \mathbb{R}^m.$$

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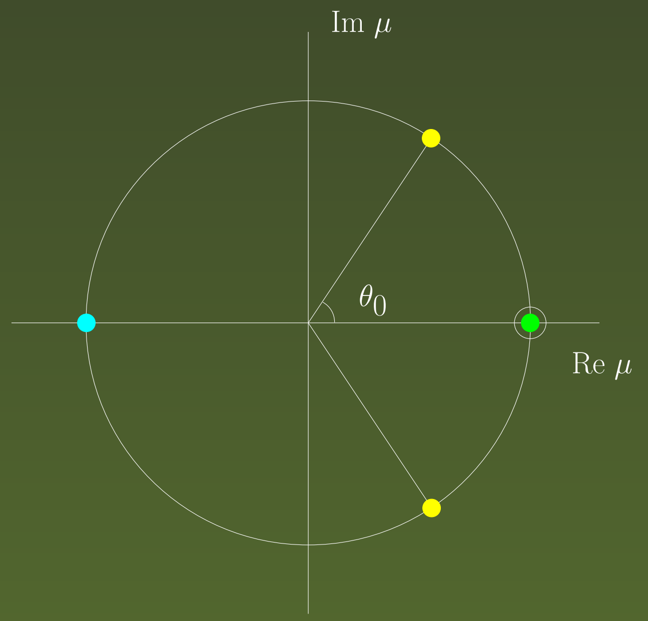
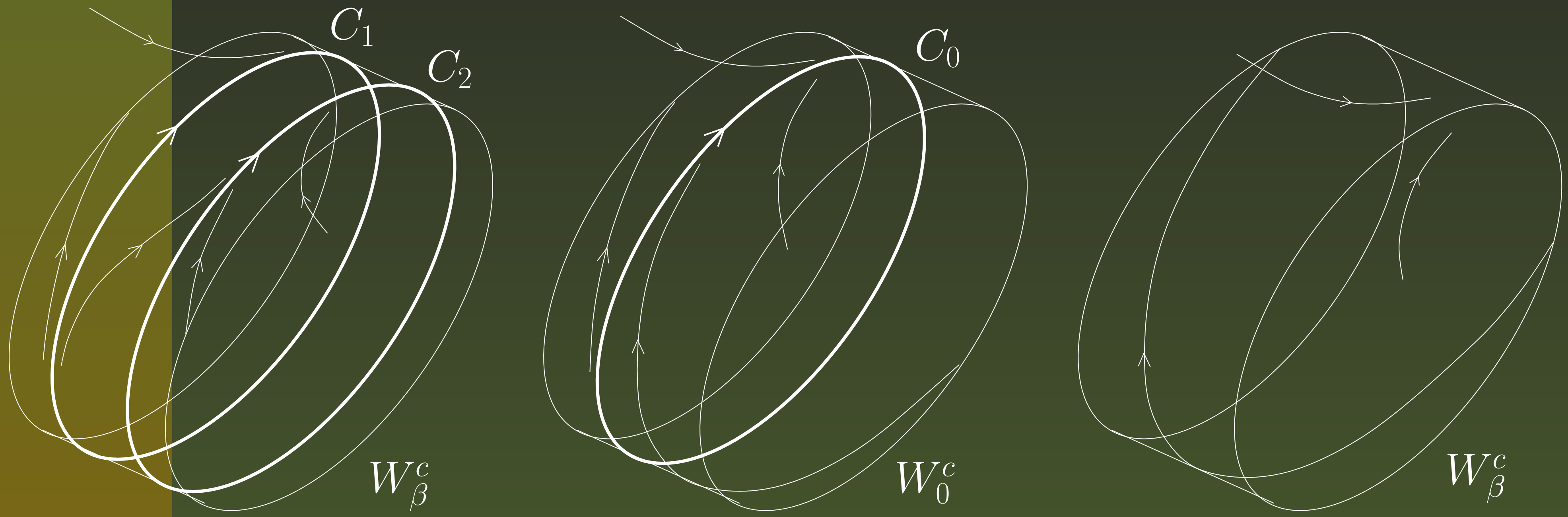
$$\dot{M}(t) - f_x(x_0(t), \alpha_0)M(t) = 0, \quad M(0) = I_n.$$

- Fold (LPC): $\mu_1 = 1$;
- Flip (PD): $\mu_1 = -1$;
- Torus (NS): $\mu_{1,2} = e^{\pm i\theta_0}$.



2.1. Generic LPC bifurcation: $\mu_1 = 1$



$\beta(\alpha) < 0$ $\beta(\alpha) = 0$ $\beta(\alpha) > 0$

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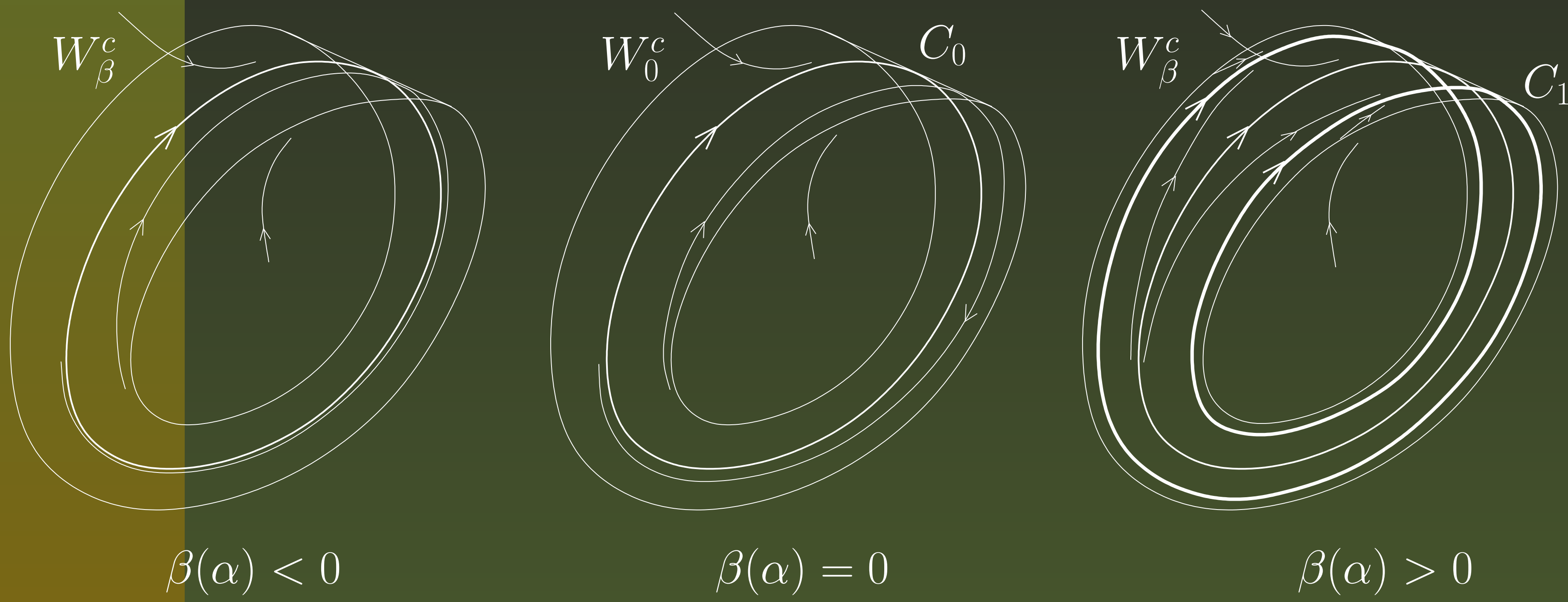
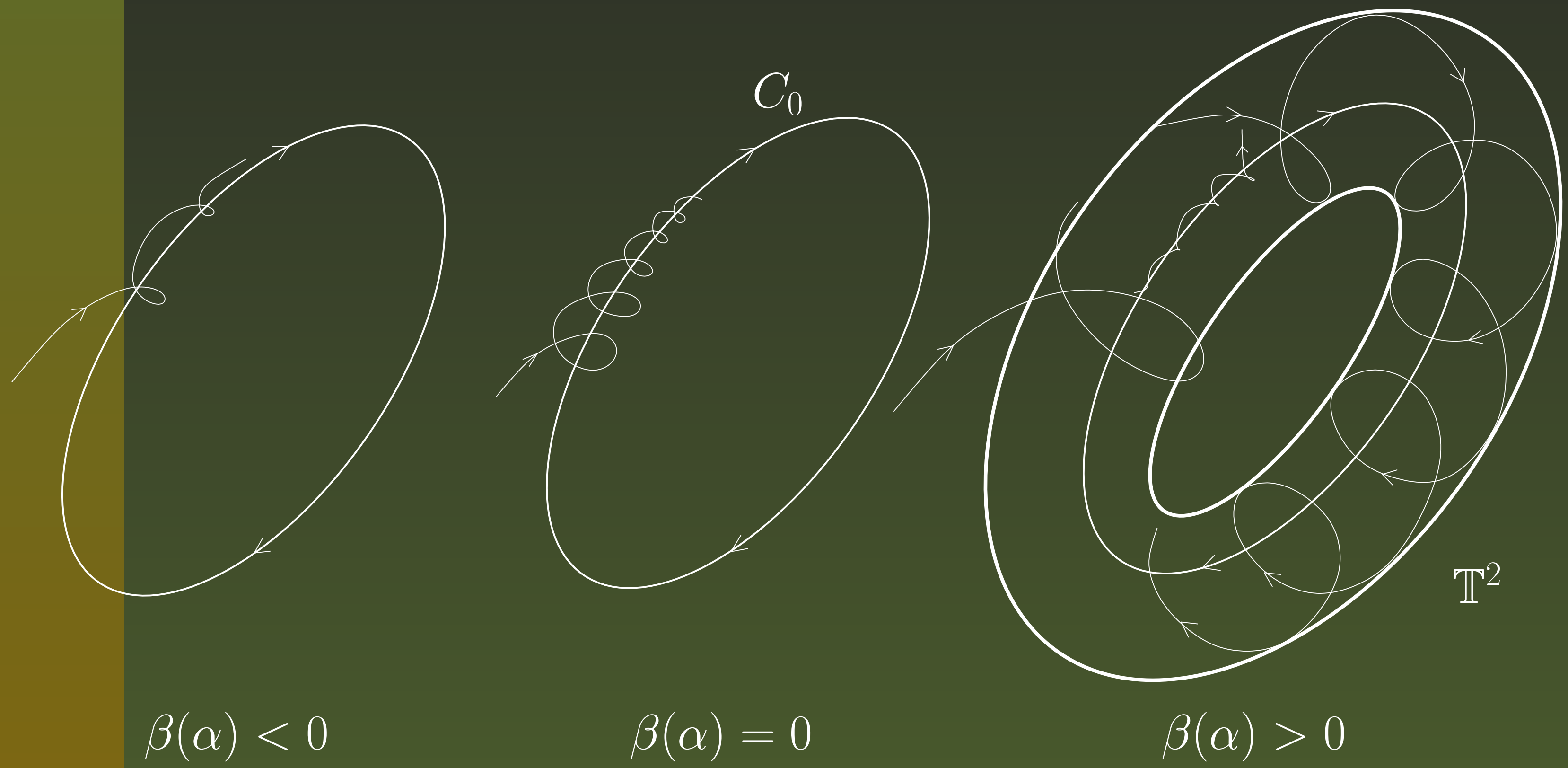
2.2. Generic PD bifurcation: $\mu_1 = -1$



2.3. Generic NS bifurcation: $\mu_{1,2} = e^{\pm i\theta_0}$



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3. Standard approach using Poincaré maps

- Poincaré map and its fixed points
- Continuation of PD and LP bifurcations of maps
- Continuation of NS bifurcation of maps
- Smooth normal forms of maps on center manifolds
- Difficulties using Poincaré maps

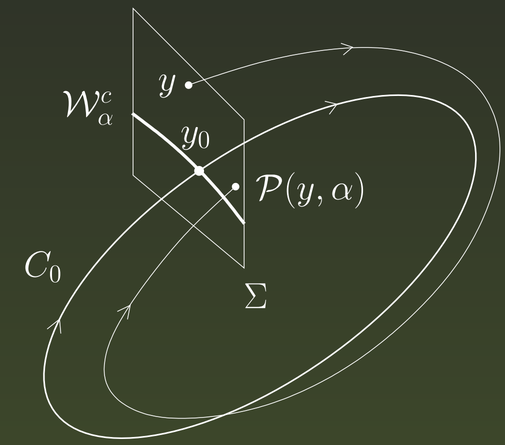
Reference on finite-dimensional bordering techniques:

- *Govaerts, W. Numerical Methods for Bifurcations of Dynamical Equilibria.* SIAM, 2000.

3.1. Poincaré map and its fixed points

- Poincaré map $\mathcal{P}: \mathbb{R}^{n-1} \times \mathbb{R}^m \to \mathbb{R}^{n-1}$
- A limit cycle C_0 corresponds to a fixed point:

$$\mathcal{P}(y_0, \alpha) - y_0 = 0.$$



- Write

$$\mathcal{P}(y_0+\eta, \alpha_0) = y_0 + \mathcal{A}\eta + \frac{1}{2}\mathcal{B}(\eta,\eta) + \frac{1}{6}\mathcal{C}(\eta,\eta,\eta) + O(\|\eta\|^4),$$

where $\mathcal{A} = \mathcal{P}_y(y_0, \alpha_0)$ and, for $i = 1, 2, \ldots, n-1$,

$$\mathcal{B}_i(\eta,\zeta) = \sum_{j,k=1}^{n-1} \left.\frac{\partial^2 \mathcal{P}_i(y,\alpha_0)}{\partial y_j \partial y_k}\right|_{y=y_0} \eta_j \zeta_k,$$

$$\mathcal{C}_i(\eta,\zeta,\omega) = \sum_{j,k,l=1}^{n-1} \left.\frac{\partial^3 \mathcal{P}_i(y,\alpha_0)}{\partial y_j \partial y_k \partial y_l}\right|_{y=y_0} \eta_j \zeta_k \omega_l.$$

3.2. Continuation of PD and LC bifurcations of maps

- Defining system: $(y, \alpha) \in \mathbb{R}^{n-1} \times \mathbb{R}^2$

$$\begin{cases} \mathcal{P}(y,\alpha) - y &= 0, \\ g(y,\alpha) &= 0, \end{cases}$$

where g is defined by solving

$$\begin{pmatrix} \mathcal{P}_y(y,\alpha) \pm I_{n-1} & w_1 \\ v_1^{\mathrm{T}} & 0 \end{pmatrix} \begin{pmatrix} v \\ g \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- Vectors $v_1, w_1 \in \mathbb{R}^{n-1}$ are adapted to make the linear system nonsingular.
- (g_y, g_α) can be computed efficiently using the adjoint linear system.

3.3. Continuation of NS bifurcation of maps

- Defining system: $(y, \alpha, \kappa) \in \mathbb{R}^{n-1} \times \mathbb{R}^2 \times \mathbb{R}$

$$\left\{ \begin{array}{rcl} \mathcal{P}(y,\alpha) - y & = & 0, \\ g_{11}(y,\alpha,\kappa) & = & 0, \\ g_{22}(y,\alpha,\kappa) & = & 0, \end{array} \right.$$

where $\kappa = \cos\theta_0$ and g_{ij} are defined by solving

$$\begin{pmatrix} I_{n-1} - 2\kappa \mathcal{P}_y(y,\alpha) + [\mathcal{P}_y(y,\alpha)]^2 & w_1 & w_2 \\ v_1^{\mathrm{T}} & 0 & 0 \\ v_2^{\mathrm{T}} & 0 & 0 \end{pmatrix} \begin{pmatrix} r & s \\ g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

- Vectors $v_{1,2}, w_{1,2} \in \mathbb{R}^{n-1}$ are adapted to ensure unique solvability.
- Efficient computation of derivatives of g_{ij}.

3.4. Smooth normal forms of maps on center manifolds

	Eigenvectors	Normal form	Critical coefficients
LP	$\mathcal{A}q = q$ $\mathcal{A}^{\mathrm{T}}p = p$ $\langle p, q\rangle = 1$	$\xi \mapsto \beta + \xi + \tilde{b}\xi^2$ $+O(\xi^3), \ \ \xi \in \mathbb{R}$	$\tilde{b} = \frac{1}{2}\langle p, \mathcal{B}(q,q)\rangle$
PD	$\mathcal{A}q = -q$ $\mathcal{A}^{\mathrm{T}}p = -p$ $\langle p, q\rangle = 1$	$\xi \mapsto -(1+\beta)\xi + \tilde{c}\xi^3$ $+O(\xi^4), \ \ \xi \in \mathbb{R}$	$\tilde{c} = \frac{1}{6}\langle p, \mathcal{C}(q,q,q) + 3\mathcal{B}(q,h_2)\rangle$ $h_2 = (I_n - A)^{-1}\mathcal{B}(q,q)$
NS	$\mathcal{A}q = \mathrm{e}^{i\theta_0}q$ $\mathcal{A}^{T}p = \mathrm{e}^{-i\theta_0}p$ $\mathrm{e}^{i\nu\theta_0} \neq 1$ $\nu = 1, 2, 3, 4$ $\langle p, q\rangle = 1$	$\xi \mapsto \xi \mathrm{e}^{i\theta}\left(1 + \beta + \tilde{d}\lvert\xi\rvert^2\right)$ $+ O(\lvert\xi\rvert^4), \ \ \xi \in \mathbb{C}$	$\tilde{d} = \frac{1}{2}\mathrm{e}^{-i\theta_0}\langle p, \mathcal{C}(q,q,\bar{q})$ $+ 2\mathcal{B}(q,h_{11})$ $+ \mathcal{B}(\bar{q},h_{20})\rangle$ $h_{11} = (I_n - \mathcal{A})^{-1}\mathcal{B}(q,\bar{q})$ $h_{20} = (\mathrm{e}^{2i\theta_0}I_n - \mathcal{A})^{-1}\mathcal{B}(q,q)$

LP-coefficient $\tilde{b}$

Assume $y_0 = 0$ and write

$$\mathcal{P}(H) = \mathcal{A}H + \frac{1}{2}\mathcal{B}(H,H) + O(\|H\|^3),$$

and locally represent the center manifold $\mathcal{W}_0^c$ as the graph of a function $H : \mathbb{R} \to \mathbb{R}^{n-1}$,

$$u = H(\xi) = \xi q + \frac{1}{2}h_2\xi^2 + O(\xi^3),\ \ \xi \in \mathbb{R},\ h_2 \in \mathbb{R}^{n-1}.$$

The restriction of the Poincaré map to $\mathcal{W}_0^c$ is

$$\xi \mapsto \mathcal{G}(\xi) = \xi + \tilde{b}\xi^2 + O(\xi^3).$$

The invariance of the center manifold $\mathcal{W}_0^c$ means

$$\mathcal{P}(H(\xi)) = H(\mathcal{G}(\xi)).$$

$$\mathcal{A}(\xi q + \frac{1}{2}h_2\xi^2) + \frac{1}{2}\mathcal{B}(\xi q, \xi q) + O(|\xi|^3) = (\xi + \tilde{b}\xi^2)q + \frac{1}{2}h_2\xi^2 + O(|\xi|^3)$$

- The ξ-terms give the identity: $\mathcal{A}q = q$.
- The ξ^2-terms give the equation for h_2:

$$(\mathcal{A} - I_{n-1})h_2 = -\mathcal{B}(q, q) + 2\tilde{b}q.$$

It is singular and its Fredholm solvability implies

$$\tilde{b} = \frac{1}{2}\langle p, \mathcal{B}(q, q)\rangle,$$

where $\mathcal{A}^{\mathrm{T}}p = p$, $\langle p, q\rangle = 1$.

3.5. Difficulties using Poincaré maps

- The computation of $\mathcal{P}$ can be unstable numerically near saddle or repelling limit cycles.
- Finite differences give low accuracy for $\mathcal{B}$ and $\mathcal{C}$ even if the cycle C_0 is stable.
- Simultaneous solving variational equations for the components of $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ is possible but costly and difficult to implement.

4. Continuation and normal form analysis using BVPs

- Continuation of limit cycles in one parameter
- Simple bifurcation points
- Continuation of bifurcations in two parameters
- Periodic normalization on center manifolds

4.1. Continuation of limit cycles in one parameter

- Defining system (BVP with IC) [Keller & Doedel, 1981]:

$$\left\{\begin{array}{rcl} \dot{u}(\tau) - Tf(u(\tau),\alpha) & = & 0, \;\; \tau \in [0,1], \\ u(0) - u(1) & = & 0, \\ \int_0^1 \langle \dot{\tilde{u}}(\tau), u(\tau) \rangle \, d\tau & = & 0, \end{array}\right.$$

where $\tilde{u}$ is a reference periodic solution.

- Linearization w.r.t. (u, T, α):

$$\left[\begin{array}{ccc} D - Tf_x(u,\alpha) & -f(u,\alpha) & -Tf_\alpha(u,\alpha) \\ \delta_0 - \delta_1 & 0 & 0 \\ \mathrm{Int}_{\dot{\tilde{u}}} & 0 & 0 \end{array}\right]$$

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Discretization via orthogonal collocation

- Mesh points: $0 = \tau_0 < \tau_1 < \cdots < \tau_N = 1$.
- Basis points:

$$\tau_{i,j} = \tau_i + \frac{j}{m}(\tau_{i+1} - \tau_i),\ i = 0, 1, \ldots, N-1,\ j = 0, 1, \ldots, m.$$

- Approximation: $u^{(i)}(\tau) = \sum_{j=0}^{m} u^{i,j} l_{i,j}(\tau),\ \ \tau \in [\tau_i, \tau_{i+1}]$, where $l_{i,j}(\tau)$ are the Lagrange basis polynomials and $u^{i,m} = u^{i+1,0}$.
- Collocation [de Boor & Swartz, 1973]:

$$\begin{cases} \left(\sum_{j=0}^{m} u^{i,j} l'_{i,j}(\zeta_{i,k})\right) - Tf(\sum_{j=0}^{m} u^{i,j} l_{i,j}(\zeta_{i,k}), \alpha) & = 0, \\ u^{0,0} - u^{N-1,m} & = 0, \\ \sum_{i=0}^{N-1} \sum_{j=0}^{m} \sigma_{i,j} \langle \dot{\tilde{u}}^{i,j}, u^{i,j} \rangle & = 0, \end{cases}$$

where $\zeta_{i,k},\ k = 1, 2, \ldots, m$, are the Gauss points.

Sparse Jacobian matrix

$$
\begin{pmatrix}
u^{0,0} & & u^{0,1} & & u^{1,0} & & u^{1,1} & & u^{2,0} & & u^{2,1} & & u^{3,0} & & T & \alpha \\
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\end{pmatrix}
$$

4.2. Simple bifurcation points

$$\begin{aligned}\dot{\Phi}(\tau) - Tf_x(u(\tau),\alpha_0)\Phi(\tau) &= 0, \quad \Phi(0) = I_n,\\ \dot{\Psi}(\tau) + Tf_x^{\mathrm{T}}(u(\tau),\alpha_0)\Psi(\tau) &= 0, \quad \Psi(0) = I_n.\end{aligned}$$

- LPC:

$$(\Phi(1)-I_n)q_0 = 0, (\Phi(1)-I_n)q_1 = q_0, (\Psi(1)-I_n)p_0 = 0, (\Psi(1)-I_n)p_1 = p_0.$$

- PD:

$$(\Phi(1) + I_n)q_2 = 0, \;\; (\Psi(1) + I_n)p_2 = 0.$$

- NS: $\kappa = \cos\theta_0$

$$(\Phi(1) - \mathrm{e}^{i\theta_0} I_n)(q_3 + iq_4) = 0, \;\; (\Psi(1) - \mathrm{e}^{-i\theta_0} I_n)(p_3 + ip_4) = 0.$$

We have $(I_n - 2\kappa\Phi(1) + \Phi^2(1))q_{3,4} = 0.$

4.3. Continuation of bifurcations in two parameters

- PD and LPC: $(u, T, \alpha) \in C^1([0,1], \mathbb{R}^n) \times \mathbb{R} \times \mathbb{R}^2$

$$\begin{cases} \dot{u}(\tau) - Tf(u(\tau), \alpha) & = & 0, \quad \tau \in [0,1], \\ u(0) - u(1) & = & 0, \\ \int_0^1 \langle \dot{\tilde{u}}(\tau), u(\tau) \rangle \, d\tau & = & 0, \\ G[u, T, \alpha] & = & 0. \end{cases}$$

- NS: $(u, T, \alpha, \kappa) \in C^1([0,1], \mathbb{R}^n) \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}$

$$\begin{cases} \dot{u}(\tau) - Tf(u(\tau), \alpha) & = & 0, \quad \tau \in [0,1], \\ u(0) - u(1) & = & 0, \\ \int_0^1 \langle \dot{\tilde{u}}(\tau), u(\tau) \rangle \, d\tau & = & 0, \\ G_{11}[u, T, \alpha, \kappa] & = & 0, \\ G_{22}[u, T, \alpha, \kappa] & = & 0. \end{cases}$$

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PD-continuation

- There exist $v_{01}, w_{01} \in C^0([0,1],\mathbb{R}^n)$, and $w_{02} \in \mathbb{R}^n$, such that $N_1 : C^1([0,1],\mathbb{R}^n) \times \mathbb{R} \to C^0([0,1],\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}$,

$$N_1 = \begin{bmatrix} D - Tf_x(u,\alpha) & w_{01} \\ \delta_0 - \delta_1 & w_{02} \\ \mathrm{Int}_{v_{01}} & 0 \end{bmatrix},$$

  is one-to-one and onto near a simple PD bifurcation point.

- Define G by solving $\quad N_1 \begin{pmatrix} v \\ G \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$

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- The BVP for (v, G) can be written in the "classical form"

$$\begin{cases} \dot{v}(\tau) - Tf_x(u(\tau), \alpha)v(\tau) + Gw_{01}(\tau) & = & 0, \ \tau \in [0,1], \\ v(0) + v(1) + Gw_{02} & = & 0, \\ \int_0^1 \langle v_{01}(\tau), v(\tau) \rangle d\tau - 1 & = & 0. \end{cases}$$

- One can take

$$w_{02} = 0$$

and

$$w_{01}(\tau) = \Psi(\tau)p_2, \ \ v_{01}(\tau) = \Phi(\tau)q_2.$$

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LPC-continuation

- There exist $v_{01}, w_{01} \in C^0([0,1],\mathbb{R}^n), w_{02} \in \mathbb{R}^n$, and $v_{02}, w_{03} \in \mathbb{R}$ such that $N_2 : C^1([0,1],\mathbb{R}^n) \times \mathbb{R}^2 \to C^0([0,1],\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^2$,

$$N_2 = \begin{bmatrix} D - Tf_x(u,\alpha) & -f(u,\alpha) & w_{01} \\ \delta_0 - \delta_1 & 0 & w_{02} \\ \mathrm{Int}_{f(u,\alpha)} & 0 & w_{03} \\ \mathrm{Int}_{v_{01}} & v_{02} & 0 \end{bmatrix},$$

is one-to-one and onto near a simple LPC bifurcation point.

- Define G by solving $\quad N_2 \begin{pmatrix} v \\ S \\ G \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$

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NS-continuation

- There exist $v_{01}, v_{02}, w_{11}, w_{12} \in C^0([0,2],\mathbb{R}^n)$, and $w_{21}, w_{22} \in \mathbb{R}^n$, such that $N_3 : C^1([0,2],\mathbb{R}^n) \times \mathbb{R}^2 \to C^0([0,2],\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^2$,

$$N_3 = \begin{bmatrix} D - Tf_x(u,\alpha) & w_{11} & w_{12} \\ \delta_0 - 2\kappa\delta_1 + \delta_2 & w_{21} & w_{22} \\ \mathrm{Int}_{v_{01}} & 0 & 0 \\ \mathrm{Int}_{v_{02}} & 0 & 0 \end{bmatrix},$$

  is one-to-one and onto near a simple NS bifurcation point.

- Define G_{jk} by solving $N_3 \begin{pmatrix} r & s \\ G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$

Remarks on continuation of bifurcations

- After discretization via orthogonal collocation, all linear BVPs for G's have sparsity structure that is identical to that of the linearization of the BVP for limit cycles.
- For each defining system holds: *Simplicity of the bifurcation + Transversality* $\Rightarrow$ *Regularity of the defining BVP*.
- Jacobian matrix of each (discretized) defining BVP can be efficiently computed using adjoint linear BVP.
- Border adaptation using solutions of the adjoint linear BVPs.
- Actually implemented in MATCONT, also with compiled C-codes for the Jacobian matrices.

4.4. Periodic normalization on center manifolds

- Parameter-dependent periodic normal forms for LPC, PD, and NS
- Critical normal form coefficients

References on periodic normal forms:

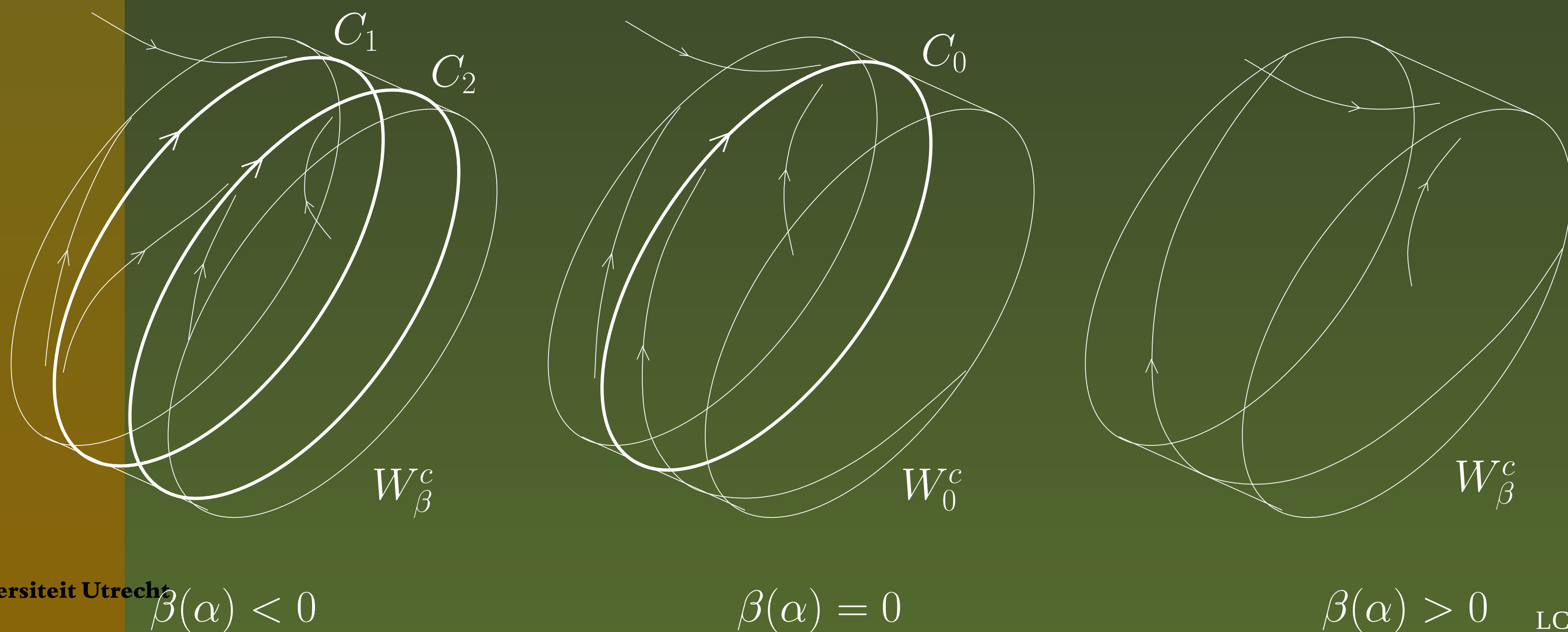
- *Iooss, G.* Global characterization of the normal form for a vector field near a closed orbit. *J. Diff. Equations* **76** (1988), 47-76.
- *Iooss, G. and Adelmeyer, M. Topics in Bifurcation Theory and Applications*. World Scientific, 1992.

LPC bifurcation

T_0-periodic normal form on W^c_α:

$$\begin{cases} \dfrac{d\tau}{dt} &= 1+\nu(\alpha)-\xi+a(\alpha)\xi^2+\mathcal{O}(\xi^3), \\ \dfrac{d\xi}{dt} &= \beta(\alpha)+b(\alpha)\xi^2+\mathcal{O}(\xi^3), \end{cases}$$

where $a, b \in \mathbb{R}$, $\text{sign}(b) = \text{sign}(\tilde{b})$.



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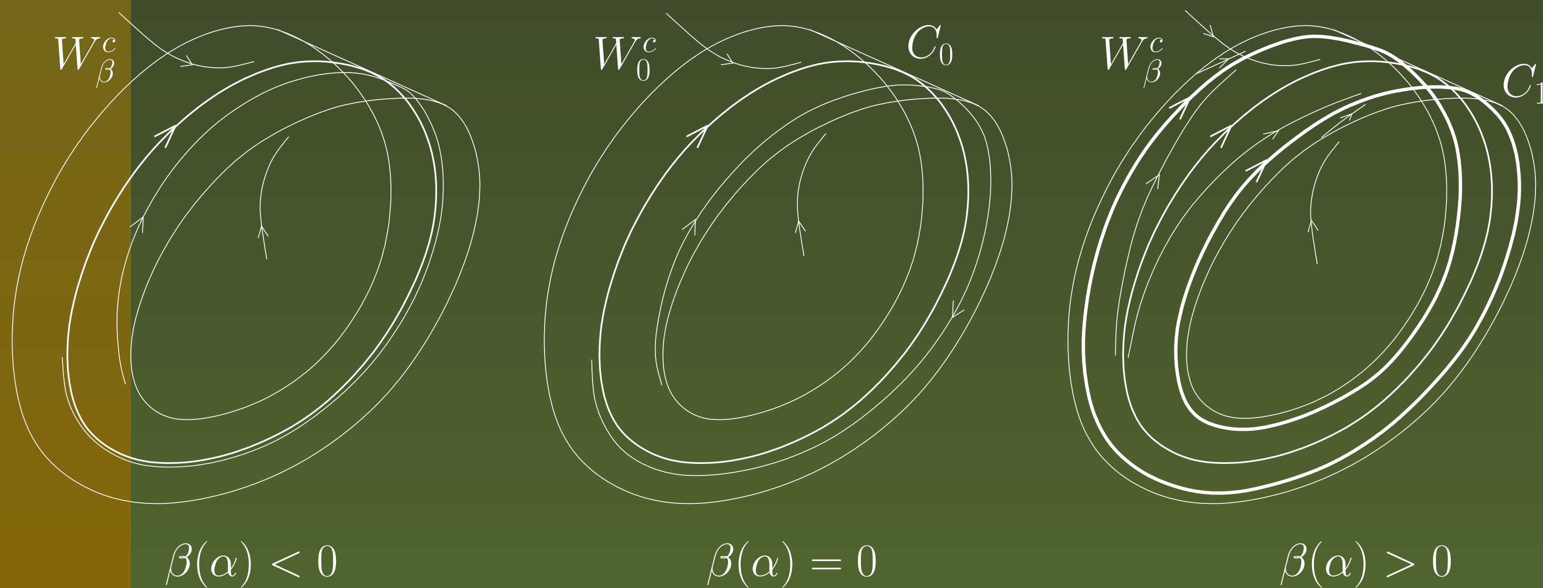
PD bifurcation

$2T_0$-periodic normal form on W^c_α:

$$\begin{cases} \dfrac{d\tau}{dt} = 1 + \nu(\alpha) + a(\alpha)\xi^2 + \mathcal{O}(\xi^4), \\ \dfrac{d\xi}{dt} = \beta(\alpha)\xi + c(\alpha)\xi^3 + \mathcal{O}(\xi^4), \end{cases}$$

where $a, c \in \mathbb{R}$, $\text{sign}(c) = -\text{sign}(\tilde{c})$.



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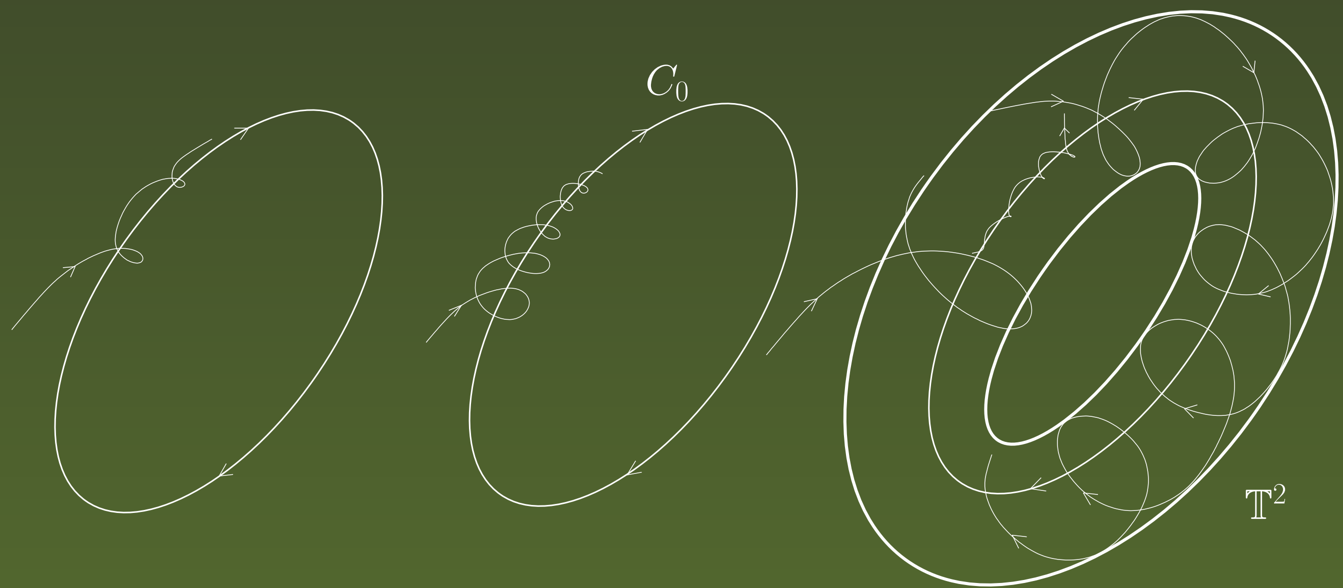
NS bifurcation

T_0-periodic normal form on W^c_α:

$$\left\{\begin{array}{rcl} \dfrac{d\tau}{dt} & = & 1+\nu(\alpha)+a(\alpha)|\xi|^2+\mathcal{O}(|\xi|^4), \\ \dfrac{d\xi}{dt} & = & \left(\beta(\alpha)+\dfrac{i\theta(\alpha)}{T(\alpha)}\right)\xi+d(\alpha)\xi|\xi|^2+\mathcal{O}(|\xi|^4), \end{array}\right.$$

where $a\in\mathbb{R}, d\in\mathbb{C}$, $\text{sign}(\text{Re}(d))=\text{sign}(\text{Re}(\tilde{d}))$.



$\beta(\alpha)<0$ $\beta(\alpha)=0$ $\beta(\alpha)>0$

Critical normal form coefficients

At a codimension-one point write

$$f(x_0(t)+v,\alpha_0) = f(x_0(t),\alpha_0)+A(t)v+\frac{1}{2}B(t;v,v)+\frac{1}{6}C(t;v,v,v)+O(\|v\|^4),$$

where $A(t) = f_x(x_0(t),\alpha_0)$ and the components of the multilinear functions B and C are given by

$$B_i(t;u,v) = \sum_{j,k=1}^{n} \left.\frac{\partial^2 f_i(x,\alpha_0)}{\partial x_j \partial x_k}\right|_{x=x_0(t)} u_j v_k$$

and

$$C_i(t;u,v,w) = \sum_{j,k,l=1}^{n} \left.\frac{\partial^3 f_i(x,\alpha_0)}{\partial x_j \partial x_k \partial x_l}\right|_{x=x_0(t)} u_j v_k w_l,$$

for $i = 1, 2, \ldots, n$. These are T_0-periodic in t.

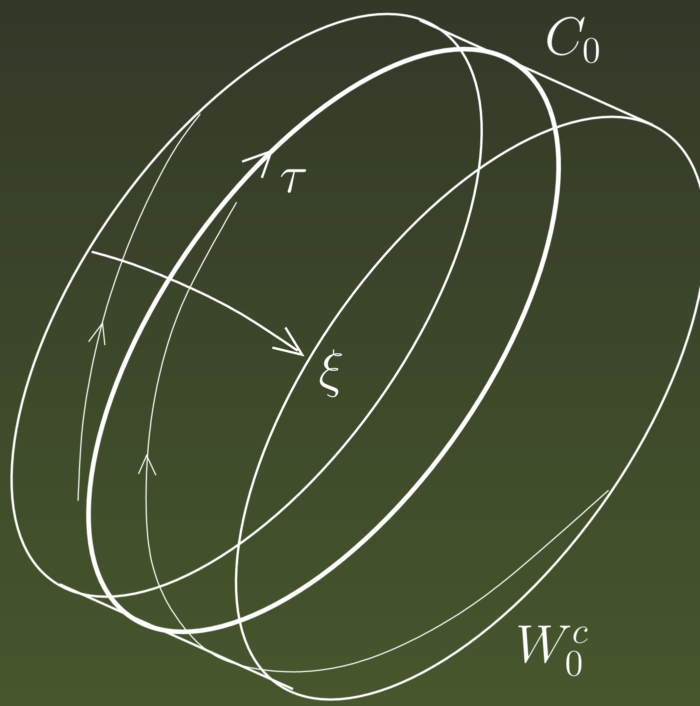
Fold (LPC): $\mu_1 = 1$

- Critical center manifold $W_0^c : \tau \in [0, T_0],\ \xi \in \mathbb{R}$

$$x = x_0(\tau) + \xi v(\tau) + H(\tau, \xi),$$

where $H(T_0, \xi) = H(0, \xi)$,

$$H(\tau, \xi) = \frac{1}{2} h_2(\tau)\xi^2 + O(\xi^3)$$



- Critical periodic normal form on W_0^c:

$$\begin{cases} \dfrac{d\tau}{dt} = 1 - \xi + a\xi^2 + \mathcal{O}(\xi^3), \\ \dfrac{d\xi}{dt} = b\xi^2 + \mathcal{O}(\xi^3), \end{cases}$$

where $a, b \in \mathbb{R}$, while the $\mathcal{O}(\xi^3)$-terms are T_0-periodic in τ.

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LPC: Eigenfunctions

$$\left\{\begin{array}{rcl} \dot{v}(\tau) - A(\tau)v(\tau) - f(x_0(\tau),\alpha_0) & = & 0,\ \tau \in [0,T_0], \\ v(0) - v(T_0) & = & 0, \\ \int_0^{T_0} \langle v(\tau), f(x_0(\tau),\alpha_0)\rangle d\tau & = & 0, \end{array}\right.$$

implying

$$\int_0^{T_0} \langle \varphi^*(\tau), f(x_0(\tau),\alpha_0)\rangle \, d\tau = 0,$$

where φ^* satisfies

$$\left\{\begin{array}{rcl} \dot{\varphi}^*(\tau) + A^{\mathrm{T}}(\tau)\varphi^*(\tau) & = & 0,\ \tau \in [0,T_0], \\ \varphi^*(0) - \varphi^*(T_0) & = & 0, \\ \int_0^{T_0} \langle \varphi^*(\tau), v(\tau)\rangle d\tau - 1 & = & 0. \end{array}\right.$$

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LPC: Computation of b

- Substitute into

$$\frac{dx}{dt} = \frac{\partial x}{\partial \xi}\frac{d\xi}{dt} + \frac{\partial x}{\partial \tau}\frac{d\tau}{dt}$$

- Collect

$$\begin{aligned}
\xi^0 &: \quad \dot{x}_0 = f(x_0, \alpha_0), \\
\xi^1 &: \quad \dot{v} - A(\tau)v = \dot{x}_0, \\
\xi^2 &: \quad \dot{h}_2 - A(\tau)h_2 = B(\tau; v, v) - 2af(x_0, \alpha_0) + 2\dot{v} - 2bv.
\end{aligned}$$

- Fredholm solvability condition

$$b = \frac{1}{2}\int_0^{T_0} \langle \varphi^*(\tau), B(\tau; v(\tau), v(\tau)) + 2A(\tau)v(\tau)\rangle \, d\tau.$$

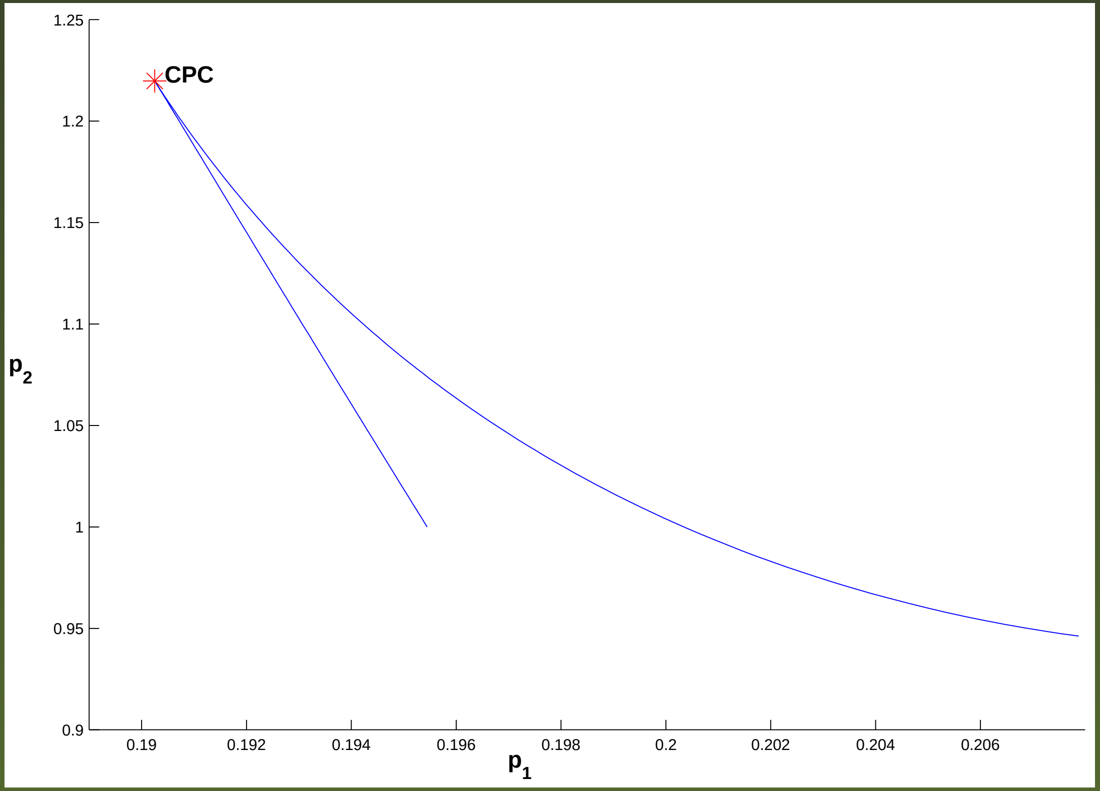
LPC: Example in MATCONT (ABC-reaction model)

$$\begin{cases} \dot{u}_1 = -u_1 + p_1(1-u_1)e^{u_3}, \\ \dot{u}_2 = -u_2 + p_1(1-u_1-p_5u_2)e^{u_3}, \\ \dot{u}_3 = -u_3 - p_3u_3 + p_1p_4(1-u_1+p_2p_5u_2)e^{u_3}. \end{cases}$$

$p_3 = 1.5$

$p_4 = 8.0$

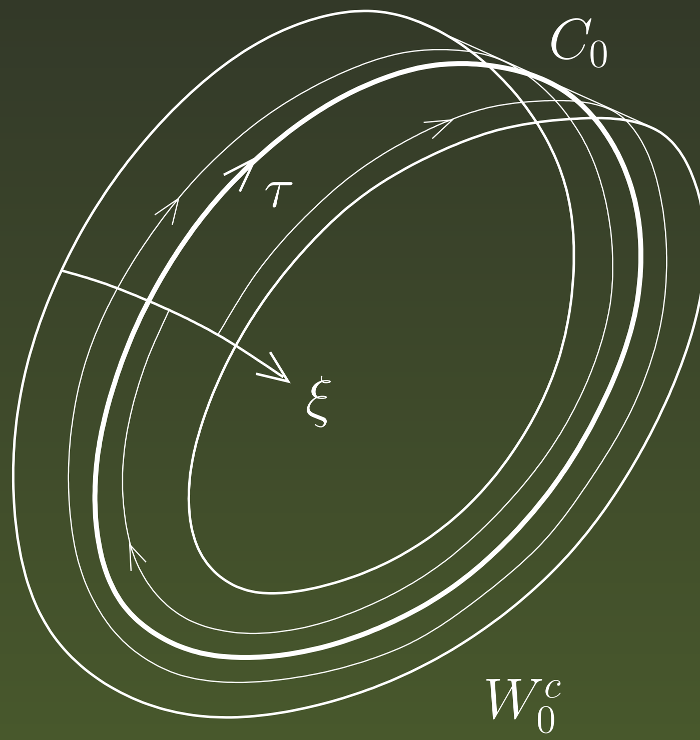
$p_5 = 0.04$



$b = 0$ at CPC (cusp)

Flip (PD): $\mu_1 = -1$

- Critical center manifold $W_0^c : \tau \in [0, 2T_0],\ \xi \in \mathbb{R}$



$$x = x_0(\tau) + \xi w(\tau) + H(\tau, \xi),$$

where $H(2T_0, \xi) = H(0, \xi)$,

$$H(\tau, \xi) = \frac{1}{2} h_2(\tau)\xi^2 + \frac{1}{6} h_3(\tau)\xi^3 + O(\xi^4)$$

- Critical periodic normal form on W_0^c:

$$\begin{cases} \dfrac{d\tau}{dt} &= 1 + a\xi^2 + \mathcal{O}(\xi^4), \\ \dfrac{d\xi}{dt} &= c\xi^3 + \mathcal{O}(\xi^4), \end{cases}$$

where $a, c \in \mathbb{R}$, while the $\mathcal{O}(\xi^4)$-terms are $2T_0$-periodic in τ.

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PD: Eigenfunctions

$$w(\tau) = \begin{cases} v(\tau), & \tau \in [0, T_0], \\ -v(\tau - T_0), & \tau \in [T_0, 2T_0], \end{cases} \quad w^*(\tau) = \begin{cases} v^*(\tau), & \tau \in [0, T_0], \\ -v^*(\tau - T_0), & \tau \in [T_0, 2T_0], \end{cases}$$

with

$$\begin{cases} \dot{v}(\tau) - A(\tau)v(\tau) & = & 0, \ \tau \in [0, T_0], \\ v(0) + v(T_0) & = & 0, \\ \int_0^{T_0} \langle v(\tau), v(\tau) \rangle d\tau - 1 & = & 0, \end{cases}$$

$$\begin{cases} \dot{v}^*(\tau) + A^{\mathrm{T}}(\tau)v^*(\tau) & = & 0, \ \tau \in [0, T_0], \\ v^*(0) + v^*(T_0) & = & 0, \\ \int_0^{T_0} \langle v^*(\tau), v(\tau) \rangle d\tau - 1/2 & = & 0. \end{cases}$$

PD: Quadratic terms

$$\xi^2 \;:\; \dot{h}_2 - A(\tau)h_2 = B(\tau; w, w) - 2a\dot{x}_0, \quad \tau \in [0, 2T_0].$$

Since $\mathrm{Ker}\left(\frac{d}{d\tau} - A(\tau)\right) = \mathrm{span}\{w, \psi = \dot{x}_0\}$, we must have

$$\begin{cases} \int_0^{2T_0} \langle w^*(\tau), B(\tau; w(\tau), w(\tau)) - 2a\dot{x}_0(\tau)\rangle \, d\tau & = & 0, \\ \int_0^{2T_0} \langle \psi^*(\tau), B(\tau; w(\tau), w(\tau)) - 2a\dot{x}_0(\tau)\rangle \, d\tau & = & 0, \end{cases}$$

where ψ^* satisfies

$$\begin{cases} \dot{\psi}^*(\tau) + A^{\mathrm{T}}(\tau)\psi^*(\tau) & = & 0, \; \tau \in [0, T_0], \\ \psi^*(0) - \psi^*(T_0) & = & 0, \\ \int_0^{T_0} \langle \psi^*(\tau), f(x_0(\tau), \alpha_0)\rangle \, d\tau - 1/2 & = & 0, \end{cases}$$

and is extended to $[T_0, 2T_0]$ by periodicity.

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PD: Computation of a and h_2

- The first Fredholm condition holds identically for all a, while the second gives

$$a = \frac{1}{2}\int_0^{2T_0} \langle \psi^*(\tau), B(\tau; w(\tau), w(\tau)\rangle d\tau = \int_0^{T_0} \langle \psi^*(\tau), B(\tau; v(\tau), v(\tau)\rangle d\tau.$$

- Define h_2 on $[0, T_0]$ as the unique solution to

$$\left\{ \begin{array}{rcl} \dot{h}_2(\tau) - A(\tau)h_2(\tau) - B(\tau; v(\tau), v(\tau)) + 2af(x_0(\tau), \alpha_0) & = & 0, \\ h_2(0) - h_2(T_0) & = & 0, \\ \int_0^{T_0} \langle \psi^*(\tau), h_2(\tau)\rangle \, d\tau & = & 0, \end{array} \right.$$

and extend it by periodicity to $[T_0, 2T_0]$.

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PD: Computation of c

Cubic terms:

$$\xi^3 \;:\; \dot{h}_3 - A(\tau)h_3 = C(\tau; w, w, w) + 3B(\tau; w, h_2) - 6a\dot{w} - 6cw.$$

The Fredholm solvability condition implies

$$\begin{aligned} 6c &= \int_0^{2T_0} \langle w^*(\tau), C(\tau; w(\tau), w(\tau), w(\tau)) + 3B(\tau; w(\tau), h_2(\tau))\rangle d\tau \\ &- \int_0^{2T_0} \langle w^*(\tau), 6aA(\tau)w(\tau)\rangle d\tau \end{aligned}$$

or

$$c = \frac{1}{3}\int_0^{T_0} \langle v^*(\tau), C(\tau; v(\tau), v(\tau), v(\tau)) + 3B(\tau; v(\tau), h_2(\tau)) - 6aA(\tau)v(\tau)\rangle d\tau$$

Torus (NS): $\mu_{1,2} = e^{\pm i\theta_0}$ with $e^{i\nu\theta_0} \neq 1, \nu = 1,2,3,4$

- Critical center manifold $W_0^c : \tau \in [0, T_0],\ \xi \in \mathbb{C}$

$$x = x_0(\tau) + \xi v(\tau) + \bar{\xi}\bar{v}(\tau) + H(\tau, \xi, \bar{\xi}), \quad H(T_0, \xi, \bar{\xi}) = H(0, \xi, \bar{\xi}),$$

$$H(\tau, \xi, \bar{\xi}) = \frac{1}{2} h_{20}(\tau)\xi^2 + h_{11}(\tau)\xi\bar{\xi} + \frac{1}{2} h_{02}(\tau)\bar{\xi}^2 + \frac{1}{6} h_{30}(\tau)\xi^3 + \frac{1}{2} h_{21}(\tau)\xi^2\bar{\xi} + \frac{1}{2} h_{12}(\tau)\xi\bar{\xi}^2 + \frac{1}{6} h_{03}(\tau)\bar{\xi}^3 + O(|\xi|^4).$$

- Critical periodic normal form on W_0^c:

$$\begin{cases} \dfrac{d\tau}{dt} = 1 + a|\xi|^2 + \mathcal{O}(|\xi|^4), \\ \dfrac{d\xi}{dt} = \dfrac{i\theta_0}{T_0}\xi + d\xi|\xi|^2 + \mathcal{O}(|\xi|^4), \end{cases}$$

where $a \in \mathbb{R}$, $d \in \mathbb{C}$, and the $\mathcal{O}(|\xi|^4)$-terms are T_0-periodic in τ.

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NS: Complex eigenfunctions

$$\left\{\begin{array}{rcl} \dot{v}(\tau) - A(\tau)v(\tau) + \dfrac{i\theta_0}{T_0}v(\tau) & = & 0,\ \tau \in [0, T_0], \\ v(0) - v(T_0) & = & 0, \\ \int_0^{T_0} \langle v(\tau), v(\tau)\rangle d\tau - 1 & = & 0. \end{array}\right.$$

and

$$\left\{\begin{array}{rcl} \dot{v}^*(\tau) + A^{\mathrm{T}}(\tau)v^*(\tau) - \dfrac{i\theta_0}{T_0}v^*(\tau) & = & 0,\ \tau \in [0, T_0], \\ v^*(0) - v^*(T_0) & = & 0, \\ \int_0^{T_0} \langle v^*(\tau), v(\tau)\rangle d\tau - 1 & = & 0. \end{array}\right.$$

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NS: Quadratic terms

- $\xi^2\bar{\xi}^0 \;:\; \dot{h}_{20} - A(\tau)h_{20} + \frac{2i\theta_0}{T_0}h_{20} = B(\tau; v, v)$

  Since $\mathrm{e}^{2i\theta_0}$ is not a multiplier of the critical cycle, the BVP

$$\left\{ \begin{array}{rcl} \dot{h}_{20}(\tau) - A(\tau)h_{20}(\tau) + \dfrac{2i\theta_0}{T_0}h_{20}(\tau) - B(\tau; v(\tau), v(\tau)) & = & 0, \\ h_{20}(0) - h_{20}(T_0) & = & 0. \end{array} \right.$$

  has a unique solution on $[0, T_0]$.

- $|\xi|^2 \;:\; \dot{h}_{11} - A(\tau)h_{11} = B(\tau; v, \bar{v}) - a\dot{x}_0$

  Here

$$\mathrm{Ker}\left(\frac{d}{d\tau} - A(\tau)\right) = \mathrm{span}(\varphi = \dot{x}_0).$$

NS: Computation of a and h_{11}

- Define φ^* as the unique solution of

$$\begin{cases} \dot{\varphi}^*(\tau) + A^{\mathrm{T}}(\tau)\varphi^*(\tau) &= 0,\ \tau \in [0, T_0], \\ \varphi^*(0) - \varphi^*(T_0) &= 0, \\ \int_0^{T_0} \langle \varphi^*(\tau), f(x_0(\tau), \alpha_0)\rangle d\tau - 1 &= 0. \end{cases}$$

- Fredholm solvability: $a = \int_0^{T_0} \langle \varphi^*(\tau), B(\tau; v(\tau), \bar{v}(\tau)\rangle\, d\tau$.
- Then find h_{11} on $[0, T_0]$ from the BVP

$$\begin{cases} \dot{h}_{11}(\tau) - A(\tau)h_{11}(\tau) - B(\tau; v(\tau), \bar{v}(\tau)) + af(x_0(\tau), \alpha_0) &= 0, \\ h_{11}(0) - h_{11}(T_0) &= 0, \\ \int_0^{T_0} \langle \varphi^*(\tau), h_{11}(\tau)\rangle\, d\tau &= 0. \end{cases}$$

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NS: Computation of d

- Cubic terms:

$$\xi^2\bar{\xi} \;:\; \dot{h}_{21} - Ah_{21} + \frac{i\theta_0}{T_0}h_{21} = 2B(\tau;h_{11},v) + B(\tau;h_{20},\bar{v}) + C(\tau;v,v,\bar{v}) - 2a\dot{v} - 2dv.$$

- Fredholm solvability condition:

$$d = \frac{1}{2}\int_0^{T_0} \langle v^*(\tau), C(\tau;v(\tau),v(\tau),\bar{v}(\tau))\rangle\, d\tau + \frac{1}{2}\int_0^{T_0} \langle v^*(\tau), B(\tau;h_{11}(\tau),v(\tau)) + B(\tau;h_{20}(\tau),\bar{v}(\tau))\rangle\, d\tau - a\int_0^{T_0}\langle v^*(\tau), A(\tau)v(\tau)\rangle\, d\tau + \frac{ia\theta_0}{T_0}.$$

Remarks on numerical periodic normalization

- Only the derivatives of $f(x, \alpha_0)$ are used, not those of the Poincaré map $\mathcal{P}(y, \alpha_0)$.
- Detection of codim 2 points is easy.
- After discretization via orthogonal collocation, all linear BVPs involved have the standard sparsity structure.
- One can re-use solutions to linear BVPs appearing in the continuation to compute the normal form coefficients.
- Actually implemented in MATCONT.

5. Open problems

- Automatic differentiation of the Poincaré map vs. BVPs.
- Periodic normal forms for codim 2 bifurcations of limit cycles.
- Branch switching at codim 2 points to limit cycle codim 1 continuation.