1(a) Move to a clean directory and load the HomCont demo `cir`. Follow all the steps described in the manual to compute the curve of homoclinic orbits in two parameters and detect various codimension-two homoclinic bifurcation points.

(b) At which point in parameter space does the transition from a tame to a chaotic homoclinic bifurcation occur? Note that in order to see the eigenvalues of the equilibrium in `fort.9` (`d.*`) you will need to make a change in the constant files so that `IID=3`. From those eigenvalues we note that the equilibrium (before the Hopf bifurcation) is a saddle or a saddle-focus with a one-dimensional unstable manifold and two-dimensional stable manifold. Note that AUTO has auto-detected the dimensions of these manifolds by using the eigenvalues of the equilibrium point. We could have chosen to force auto to make this choice by adding `NUNSTAB=1, NSTAB=2` to `c.cir.*`

(c) Try adding the line

```
unames={1:x,2:y,3:z}, parnames={1:nu,2:beta,3:gamma,4:r,5:a3,6:b3}
```

to the `c.cir.*` file and rerun and replot the *.2 files. What do you notice?

(d) Also try decrease each of the error tolerances `EPSL EPSU` and `EPSS` by a factor of 100. Rerun. Note how the codim 2 points are detected more accurately.

(c) Restart AUTO from one of the saddle-focus homoclinic points (e.g. at label 6) switch to regular continuation of periodic orbits `IPS=2`. Allow the period `PAR(11)` to decrease while `PAR(1)` varies. Plot the output as period against `PAR(1)`. You should see a wiggly curve with lots of fold point. This shows that it is possible to switch immediately from computation of homoclinic to periodic orbits. Repeat this process from one of the tame homoclinic orbits (e.g. from label between label 10 and 11 - you will need to rerun with a new UZR function to label a point on this part of the branch). Do you notice the difference?

(d) Try now to compute each of the Belyakov-type codimension-two points (the ones from labels 5, 10 and 11 in the original run of the demo) in the three parameters ν, β and γ (`PAR(1)`, `PAR(2)` and (`PAR(3)`)). To do this we set IFIXED equal to the number of the test function ψ_i that has been zeroed (remember to remove the same label from the list `IPSI` and the parameter $i + 20$ from the list of active continuation parameters `ICP`). Plot the three separate curves on a single graph (by using `ap( )` to append the data files).

2 Now try running the demo `kpr`. This demo illustrates many different features: (i) the computation of the adjoint variational equations and the detection of an inclination flip, (ii) the computation of a central saddle-node homoclinic orbit and the switching between saddle-node and saddle homoclinic orbits, (iii) three-parameter continuation of codim 2 homoclinic orbits.

Run the demo carefully step by step, noting which constants are being changed in each run. By appending various data files try to produce a 3D version of the final figure in the manual showing the three-parameter continuations of the non-central saddle-node homoclinic orbits and the two inclination flips. Make sure all the axes are labelled with the correct variable names rather than `PAR(1)` etc.

3 Now try running the demo `she` which illustrates the computation of a branch of heteroclinic orbits. Make a 3D plot of the heteroclinic orbits as the parameter μ varies as depicted in the AUTO manual. Try continuation of the heteroclinic orbits in the other problem parameters Q, σ and ζ.

4 (**optional**) Read chapter 27 of the manual. Try running the demos. This is advanced material only recommended for expert users, on how to switch branches to compute multi-pulse homoclinic orbits in a neighbourhood of a codim 2 homoclinic bifurcation. Try running the demos. Find the reference Oldeman *et al* (2003) referred to in the manual and try to understand the theory behind the method. Try using this method to find multi-pulse homoclinic orbits in other demos.

Choose **two** of the following questions. Submit the solution as a single `.pdf` file which comprises a write up of your working within any graphical output included. The file should be no longer than 10 pages. Note that each question is intended to open-ended and it a full investigation is not required in order to obtain good marks.

1 (Written exercise) Complete the construction of a 2D Poincaré maps in the neighbourhood of a saddle-focus homoclinic obrit as sketched in lecture 1. Try to find an asymptotic expression for the existence of two-pulse homoclinic orbits that spend an intermediary time interval in a neighbourhood of the equilibrium point at the origin. You may refer to any papers or text books as you wish.

2 (The demos `cir` and `tor`). These two demos solve the same set of equations but for different parameter values. In particular $\gamma = -0.6$ in the demo `tor` but is zero in the demo `cir`. For this assignment we shall take $\gamma = -0.6$. In practical 1 we found curves of two different homoclinic orbits to a nontrivial equilibrium via following periodic orbits to large period. Find an expression for the non-trivial equilibrium and use HomCont (`IPS=9`) to switch branches to compute curves of the two homoclinic orbits in the parameters `PAR(1)` and `PAR(2)` (ν and β). Note that restarting from a periodic orbit of large period, AUTO performs a "rotation" of the orbit automatically so that the equilibrium is located at the boundary points of the orbit. Can you find any codimension-two points? Now consider the homoclinic orbits to the origin computed in the demo `cir`. Recompute the two-parameter curves in ν and β for $\gamma = -0.6$. Find the codim-two points. Try to build up an understanding of all the bifurcations that occur in this parameter plane (you may wish to compute other curves of local bifurcations using AUTO or Matcont).

3 (Return to the demo `rev`). This system of equations is Hamiltonian and conserves the first integral

$$H(u_1, u_2, u_3, u_4) = u_2 u_4 - \frac{1}{2}u_3^2 + \frac{1}{2}u_2^2 + F(u)$$

where $F(u) = \int_u f(u)$, with $f(u) = u - u^3$ for the example in the manual and $f(u) = u - 1 + \alpha u^3 + \beta u^5$ for the extension to this example that we considered in practical 1.

By re-writing the starting data in `rev.dat.*` over the full interval $[-T, T]$ (this can be achieved by setting $U(tmax + t) = RU(tmax - t)$ where $R = [1, -1, 1, -1]^T$ in the case of `rev.dat.1` and $R = [-1, 1, -1, 1]^T$ for `rev.dat.3` where in this case $tmax = 1$) and output the data to new data files `rev.dat.4` and `rev.dat.5` say. Note that the first column of the data should be the time rescaled back to [0,1] (i.e. divided by 2). Try now to compute the homoclinic orbits over the full interval by using the conserved quantity H instead of the reversibility. To do this we add $\alpha_0 \nabla H$ to the ODE system where α_0 is a new dummy parameter. Remove `IREV=[*]` from the `c.` file and double the initial value of `PAR(11)`. It is also a good idea to double `NTST`. Again try to find a homoclinic snake as P varies for fixed values of $\beta > 0$ and $\alpha \in [0, 1]$. Can you find a way to compute the asymmetric "ladder" states that bifurcate in pitchfork bifurcations close to each of the folds? (Hint: AUTO detects many such `BP` points if `ISP=2` and `ISW=-1` tells AUTO to switch branches).

4 (Travelling waves in the FitzHugh-Nagumo system). Consider the FitzHugh-Nagumo equations for a travelling wave written in the form

$$\begin{aligned}\dot{u} &= v,\\ \dot{v} &= cv + f_a(u) + w,\\ \dot{w} &= \frac{\epsilon}{c}(u - \gamma w).\end{aligned}$$

in which $f_a(u) = u(u - a)(u - 1)$. Starting at values $c = 0.21$, $a = 0.2$, $\epsilon = 0.0025$ and $\gamma = 0$ use the homotopy method to compute a homoclinic orbit to the origin. (Hint: see the demo `fnb`). Compute a curve of homoclinic orbits in the (a, b)-plane. Find two Belyakov points and continue them in three parameters, allowing ϵ to be third parameter. Hence find a critical value of ϵ at which the two Belyakov points coincide. Explain what this might mean in terms of the existence of multi-pulse orbits. Now increase γ from zero to a fixed positive value. Again compute curves in the (a, b) plane. For parameter values at which the origin has real leading eigenvalues, switch on the computation of the twistedness `ITWIST=1`. Can you find any orbit-flip or inclination-flip bifurcations?