Let m be the largest k attaining the maximum value of $P(k)$ over all $0 \leq k \leq n$. By definition of m, $P(m-1) \leq P(m) > P(m+1)$. That is,

$$m \leq np + p < m + 1$$

by the equivalence of (1) and (5) for $k = m$ and $k = m+1$. Thus m is the greatest integer less than or equal to $np + p$. (Strictly speaking, the cases $m = 0$ and $m = n$ should be considered separately, but the conclusion is the same.) □

The mean. The number np, which is always close to the mode of the binomial distribution, is called the *expected number of successes*, or the *mean* of the binomial (n, p) distribution, usually denoted μ (Greek letter mu). In case the mean μ is an integer, it turns out that μ is the most likely number of successes. But if μ is not an integer, μ is not even a possible number of successes.

Expected Number of Successes (Mean of Binomial Distribution)

$$\mu = np$$

Remark. For the time being this formula is taken as the definition of the mean of a binomial distribution. Chapter 3 gives a more general, consistent definition.

Behavior of the binomial distribution for large n. This is displayed in the last two figures. As a general rule, for large values of n, the binomial distribution concentrates on a range of values around the expected value np which, while becoming larger on an absolute numerical scale, becomes smaller on a relative scale in comparison with n. Put another way, as n increases, it becomes harder to predict the number of successes exactly, but easier to predict the proportion of successes, which will most likely be close to p. This is made more precise by the *square root law* and the *law of large numbers*, discussed in the following sections. Apart from slight variations in height and width, and some slight skewness toward the edges, all the histograms follow a bell-shaped curve of roughly the same form. This is the famous *normal curve*, first discovered by De Moivre, around 1730, as an approximation to binomial distribution for large values of n.

Exercises 2.1

1. a) How many sequences of zeros and ones of length 7 contain exactly 4 ones and 3 zeros?

b) If you roll 7 dice, what is the chance of getting exactly 4 sixes?

2. Suppose that in 4-child families, each child is equally likely to be a boy or a girl, independently of the others. Which would then be more common, 4-child families with 2 boys and 2 girls, or 4-child families with different numbers of boys and girls? What would be the relative frequencies?

3. Suppose 5 dice are rolled. Assume they are fair and the rolls are independent. Calculate the probability of the following events:
A = (exactly two sixes); B = (at least two sixes); C = (at most two sixes);
D = (exactly three dice show 4 or greater); E = (at least 3 dice show 4 or greater).

4. A die is rolled 8 times. Given that there were 3 sixes in the 8 rolls, what is the probability that there were 2 sixes in the first five rolls?

5. Given that there were 12 heads in 20 independent coin tosses, calculate

a) the chance that the first toss landed heads;

b) the chance that the first two tosses landed heads;

c) the chance that at least two of the first five tosses landed heads.

6. A man fires 8 shots at a target. Assume that the shots are independent, and each shot hits the bull's eye with probability 0.7.

a) What is the chance that he hits the bull's eye exactly 4 times?

b) Given that he hit the bull's eye at least twice, what is the chance that he hit the bull's eye exactly 4 times?

c) Given that the first two shots hit the bull's eye, what is the chance that he hits the bull's eye exactly 4 times in the 8 shots?

7. You roll a die, and I roll a die. You win if the number showing on your die is strictly greater than the one on mine. If we play this game five times, what is the chance that you win at least four times?

8. For each positive integer n, what is the largest value of p such that zero is the most likely number of successes in n independent trials with success probability p?

9. The chance of winning a bet on 00 at roulette is $1/38 = 0.026315$. In 325 bets on 00 at roulette, the chance of six wins is 0.104840. Use this fact, and consideration of odds ratios, to answer the following questions without long calculations.

a) What is the most likely number of wins in 325 bets on 00, and what is its probability?

b) Find the chance of ten wins in 325 bets on 00.

c) Find the chance of ten wins in 326 bets on 00.

10. Suppose a fair coin is tossed n times. Find simple formulae in terms of n and k for

a) $P(k-1 \text{ heads} \mid k-1 \text{ or } k \text{ heads})$;

b) $P(k \text{ heads} \mid k-1 \text{ or } k \text{ heads})$.

11. 70% of the people in a certain population are adults. A random sample of size 15 will be drawn, with replacement, from this population.

a) What is the most likely number of adults in the sample?

b) What is the chance of getting exactly this many adults?

12. A gambler decides to keep betting on red at roulette, and stop as soon as she has won a total of five bets.

a) What is the probability that she has to make exactly 8 bets before stopping?

b) What is the probability that she has to make at least 9 bets?

13. Genetics. Hereditary characteristics are determined by pairs of *genes*. A gene pair for a particular characteristic is transmitted from parents to offspring by choosing one gene at random from the mother's pair, and, independently, one at random from the father's. Each gene may have several forms, or *alleles*. For example, human beings have an allele (B) for brown eyes, and an allele (b) for blue eyes. A person with allele pair BB has brown eyes, and a person with allele pair bb has blue eyes. A person with allele pair Bb or bB will have brown eyes—the allele B is called *dominant* and b *recessive*. So to have blue eyes, one must have the allele pair bb. The alleles don't "mix" or "blend".

a) A brown-eyed (BB) woman and a blue-eyed man plan to have a child. Can the child have blue eyes?

b) A brown-eyed (Bb) woman and a blue-eyed man plan to have a child. Find the chance that the child has brown eyes.

c) A brown-eyed (Bb) woman and a brown-eyed (Bb) man plan to have a child. Find the chance that the child has brown eyes.

d) A brown-eyed woman has brown-eyed parents, both Bb. She and a blue-eyed man have a child. Given that the child has brown eyes, what is the chance that the woman carries the allele b?

14. Genetics. In certain pea plants, the allele for tallness (T) dominates over the allele for shortness (s), and the allele for purple flowers (P) dominates over the allele for white flowers (w) (see Exercise 13). According to the *principle of independent assortment*, alleles for the two characteristics (flower color and height) are chosen independently of each other.

a) A (TT, PP) plant is crossed with a (ss, ww) plant. What will the offspring look like?

b) The offspring in part a) is self-fertilized, that is, crossed with itself. Write down the possible genetic combination (of flower color and height) that the offspring of this fertilization can have, and find the chance with which each such combination occurs.

c) Ten (Ts, Pw) plants are self-fertilized, each producing a new plant. Find the chance that at least 2 of the new plants are tall with purple flowers.

15. Consider the mode m of the binomial (n, p) distribution. Use the formula $m = int\,(np + p)$ to show the following:

a) If np happens to be an integer, then $m = np$.

b) If np is not an integer, then the most likely number of successes m is one of the two integers to either side of np.

c) Show by examples that m is not necessarily the closest integer to np. Neither is m always the integer above np, nor the integer below it.