

Exercises 3.2

1. Suppose that 10% of the numbers in a list are 15, 20% of the numbers are 25, and the remaining numbers are 50. What is the average of the numbers in the list?
2. One list of 100 numbers contains 20% ones and 80% twos. A second list of 100 numbers contains 50% threes and 50% fives. A third list is obtained by taking each number in the first list and adding the corresponding number in the second list.
 - a) What is the average of the third list? Or is this not determined by the information given?
 Repeat a) with adding replaced by b) subtracting c) multiplying by d) dividing by.
3. What is the expected number of sixes appearing on three die rolls? What is the expected number of odd numbers?
4. Suppose all the numbers in a list of 100 numbers are non-negative, and the average of the list is 2. Prove that at most 25 of the numbers in the list are greater than 8.
5. In a game of Chuck-a-Luck, a player can bet \$1 on any one of the numbers 1, 2, 3, 4, 5, and 6. Three dice are rolled. If the player's number appears k times, where $k \geq 1$, the player gets $\$k$ back, plus the original stake of \$1. Otherwise, the player loses the \$1 stake. Some people find this game very appealing. They argue that they have a $1/6$ chance of getting their number on each die, so at least a $1/6 + 1/6 + 1/6 = 50\%$ chance of doubling their money. That's enough to break even, they figure, so the possible extra payoff in case their number comes up more than once puts the game in their favor.
 - a) What do you think of this reasoning?
 - b) Over the long run, how many cents per game should a player expect to win or lose playing Chuck-a-Luck?
6. Let X be the number of spades in 7 cards dealt from a well-shuffled deck of 52 cards containing 13 spades. Find $E(X)$.
7. In a circuit containing n switches, the i th switch is closed with probability p_i , $i = 1, \dots, n$. Let X be the total number of switches that are closed. What is $E(X)$? Or is it impossible to say without further assumptions?
8. Suppose $E(X^2) = 3$, $E(Y^2) = 4$, $E(XY) = 2$. Find $E[(X + Y)^2]$.
9. Let X and Y be two independent indicator random variables, with $P(X = 1) = p$ and $P(Y = 1) = r$. Find $E[(X - Y)^2]$ in terms of p and r .
10. Let A and B be independent events, with indicator random variables I_A and I_B .
 - a) Describe the distribution of $(I_A + I_B)^2$ in terms of $P(A)$ and $P(B)$.
 - b) What is $E(I_A + I_B)^2$?
11. There are 100 prize tickets among 1000 tickets in a lottery. What is the expected number of prize tickets you will get if you buy 3 tickets? What is a simple upper bound for the probability that you will win at least one prize? Compare with the actual probability. Why is the bound so close?

12. Show that if a and b are constants with $P(a \leq X \leq b) = 1$, then $a \leq E(X) \leq b$.
13. Suppose a fair die is rolled ten times. Find numerical values for the expectations of each of the following random variables:
- the sum of the numbers in the ten rolls;
 - the sum of the largest two numbers in the first three rolls;
 - the maximum number in the first five rolls;
 - the number of multiples of three in the ten rolls;
 - the number of faces which fail to appear in the ten rolls;
 - the number of different faces that appear in the ten rolls;
14. A building has 10 floors above the basement. If 12 people get into an elevator at the basement, and each chooses a floor at random to get out, independently of the others, at how many floors do you expect the elevator to make a stop to let out one or more of these 12 people?
15. **Predicting demand.** Suppose that a store buys b items in anticipation of a random demand Y , where the possible values of Y are non-negative integers y representing the number of items in demand. Suppose that each item sold brings a profit of $\$ \pi$, and each item stocked but unsold brings a loss of $\$ \lambda$. The problem is to choose b to maximize expected profit.
- Show that this problem is the same as the problem of finding the predictor b of Y which minimizes over all integers the expected loss, with loss function

$$L(y, b) = \begin{cases} -\pi y + \lambda(b - y) & \text{if } y \leq b \\ -\pi b & \text{if } y > b \end{cases}$$
 - Let $r(b) = E[L(Y, b)]$. Use calculus to show that $r(b)$ is minimized over all the real numbers b , and hence over all the integers b , at the least integer y such that $P(Y \leq y) \geq \pi/(\lambda + \pi)$. *Note.* If $\pi = \lambda$, this is the median. If $\pi/(\lambda + \pi) = k\%$, this y is called the k th percentile of the distribution of Y .
16. **Aces.** A standard deck of 52 cards is shuffled and dealt. Let X_1 be the number of cards appearing before the first ace, X_2 the number of cards between the first and second ace (not counting either ace), X_3 the number between the second and third ace, X_4 the number between the third and fourth ace, and X_5 the number after the last ace. It can be shown that each of these random variables X_i has the same distribution, $i = 1, 2, \dots, 5$, and you can assume this to be true.
- Write down a formula for $P(X_i = k)$, $0 \leq k \leq 48$.
 - Show that $E(X_i) = 9.6$. [*Hint:* Do not use your answer to a).]
 - Are X_1, \dots, X_5 pairwise independent? Prove your answer.
17. A box contains 3 red balls, 4 blue balls, and 6 green balls. Balls are drawn one-by-one without replacement until all the red balls are drawn. Let D be the number of draws made. Calculate: a) $P(D \leq 9)$; b) $P(D = 9)$; c) $E(D)$.
18. Suppose that X is a random variable with just two possible values a and b . For $x = a$ and b find a formula for $p(x) = P(X = x)$ in terms of a , b and $\mu = E(X)$.

19. A collection of tickets comes in four colors: red, blue, white, and green. There are twice as many reds as blues, equal numbers of blues and whites, and three times as many greens as whites. I choose 5 tickets at random with replacement. Let X be the number of different colors that appear.
- Find a numerical expression for $P(X \geq 4)$.
 - Find a numerical expression for $E(X)$.
20. Show that the distribution of a random variable X with possible values 0, 1, and 2 is determined by $\mu_1 = E(X)$ and $\mu_2 = E(X^2)$, by finding a formula for $P(X = x)$ in terms of μ_1 and μ_2 , $x = 0, 1, 2$.
21. **Indicators and the inclusion-exclusion formula.** Let I_A be the indicator of A . Show the following:
- the indicator of A^c , the complement of A , is $I_{A^c} = 1 - I_A$;
 - the indicator of the intersection AB of A and B is the product of I_A and I_B : $I_{AB} = I_A I_B$;
 - For any collection of events A_1, \dots, A_n , the indicator of their union is

$$I_{A_1 \cup A_2 \cup \dots \cup A_n} = 1 - (1 - I_{A_1})(1 - I_{A_2}) \cdots (1 - I_{A_n})$$
 - Expand the product in the last formula and use the rules of expectation to derive the inclusion-exclusion formula of Exercise 1.3.12.
22. **Success runs in independent trials.** Consider a sequence of $n \geq 4$ independent trials, each resulting in success (S) with probability p , and failure (F) with probability $1 - p$. Say a *run of three successes* occurs at the beginning of the sequence if the first four trials result in SSSF; a run of three successes occurs at the end of the sequence if the last four trials result in FSSS; and a run of three successes elsewhere in the sequence is the pattern FSSSF. Let $R_{3,n}$ denote the number of runs of three successes in the n trials.
- Find $E(R_{3,n})$.
 - Define $R_{m,n}$, the number of success runs of length m in n trials, similarly for $1 \leq m \leq n$. Find $E(R_{m,n})$.
 - Let R_n be the total number of non-overlapping success runs in n trials, counting runs of any length between 1 and n . Find $E(R_n)$ by using the result of b).
 - Find $E(R_n)$ another way by considering for each $1 \leq j \leq n$ the number of runs that start on the j th trial. Check that the two methods give the same answer.