# Measure and Integration Exercise 3, 2014-15

1. Consider the measurable space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, where $\mathcal{B}(\mathbb{R})$ is the Borel $\sigma$-algebra on $\mathbb{R}$. Show that any monotonically increasing or decreasing function $f : \mathbb{R} \to \mathbb{R}$ is Borel measurable i.e. $\mathcal{B}(\mathbb{R}) \setminus \mathcal{B}(\mathbb{R})$ measurable. Consider the measurable space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, where $\mathcal{B}(\mathbb{R})$ is the Borel $\sigma$-algebra on $\mathbb{R}$.

   (a) Show that any monotonically increasing or decreasing function $f : \mathbb{R} \to \mathbb{R}$ is Borel measurable i.e. $\mathcal{B}(\mathbb{R}) \setminus \mathcal{B}(\mathbb{R})$ measurable.

   (b) Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ \\ 2^{-k} & \text{if } x \in [k, k+1),\ k \in \mathbb{Z},\ k \geq 0. \end{cases}$$

   Show that $f$ is measurable, and determine the values of $\lambda(\{f > 1\})$, $\lambda(\{f < 1\}$ and $\lambda(\{1/4 \leq f < 1\}$. Find a sequence of simple functions $(f_k) \subseteq \mathcal{E}^+(\mathcal{B}(\mathbb{R}))$ such that $f_j \nearrow f$.