# Measure and Integration Solution Exercise 4, 2015-16

1. Let $(X, \mathcal{A}, \mu)$ be a measure space and $f \in \mathcal{L}^1_{\mathbb{R}}(\mu)$. Define $A_0 = \{x \in X : f(x) = 0\}$, $A_\infty = \{x \in X : |f(x)| = \infty\}$, and $A_n = \{x \in X : 1/n \leq |f(x)| < n\}$, for $n \geq 1$. Set $A = \bigcup_{n=1}^{\infty} A_n$.

   (a) Show that $\int_X |f|\,d\mu = \int_A |f|\,d\mu$. (2 pts)

   (b) Show that for every $\epsilon > 0$, there exists a positive integer $N$ such that

   $$\mu(A_N) < \infty \text{ and } \int_{A_N^c} |f|\,d\mu < \epsilon.$$

   (3 pts)

2. Let $(X, \mathcal{A}, \mu)$ be a probability space (i.e. $\mu(X) = 1$) and let $\{f_n\}$ be a sequence in $\mathcal{L}^1(\mu)$ such that $\int_X |f_n| d\mu = n$ for all $n \geq 1$. Let

   $$A_n = \{x : |f_n(x) - \int_X f_n d\mu| \geq n^3\}.$$

   (a) Show that $\mu\left(\bigcap_{m\geq 1} \bigcup_{n \geq m} A_n\right) = 0$. (2 pts)

   (b) Use part (a) to show that for every $\epsilon > 0$ there exists $m_0 \geq 1$ such that

   $$\mu\{x \in X : |f_n(x)| < n^3 + n, \text{ for all } n \geq m_0\} > 1 - \epsilon.$$

   (3 pts)