Measure and Integration Exercise 5, 2015-16

1. Consider the measure space $([0,\infty), \mathcal{B}([0,\infty)), \lambda)$, where $\mathcal{B}([0,\infty))$ is the restriction of the Borel σ-algebra on $[0,\infty)$, and λ is the restriction of Lebesgue measure on $\mathcal{B}([0,\infty))$. Define $f : [0,\infty) \to \mathbb{R}$ by

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \mathbf{1}_{[n-1,n)}(x).$$

   (a) Show that f is $\mathcal{B}([0,\infty))/\mathcal{B}(\mathbb{R})$ measurable and that $f \notin \mathcal{L}^1(\lambda)$. (2 pts)

   (b) Show that the improper Riemann-integral $(R)\int_0^\infty f(x)\,dx$ exists and is finite. Give an argument why parts (a) and (b) do not contradict Corollary 11.9. (2 pts)

2. Let $(X, \mathcal{A}, \mu)$ be a measure space, and $f \in \mathcal{L}^p_{\mathbb{R}}(\mu)$ for some $p \in [1,\infty)$. For $n \geq 1$, define $f_n = \max\Big(\min(f,n), -n\Big)$. Show that for every $\epsilon > 0$ there exists an integer $n \geq 1$ such that $\int |f - f_n|^p < \epsilon$. (3 pts)

3. Consider the measure space $([0,1], \mathcal{B}([0,1]), \lambda)$, where $\mathcal{B}([0,1])$ is the restriction of the Borel σ-algebra on $[0,1]$, and λ Lebesgue measure on $[0,1]$. Let $p \in (1,\infty)$ and let q be the conjugate of p, i.e. $\frac{1}{p} + \frac{1}{q}$. Show that if $f \in \mathcal{L}^p(\lambda)$, then

$$\lim_{n\to\infty} n^{1/q} \int_{[0,1/n]} |f|\,d\lambda = 0.$$

   (3 pts)