Measure and Integration Exercise 5, 2014-15

1. Consider the measure space $([0,1], \mathcal{B}([0,1]), \lambda)$, where $\mathcal{B}([0,1])$ is the restriction of the Borel σ-algebra on $[0,1]$, and λ Lebesgue measure on $[0,1]$.

   (a) Show that $\displaystyle\lim_{n\to\infty} \int_{[0,1]} \frac{x^n}{(1+x)^2}\, d\lambda(x) = 0$.

   (b) Show that $\displaystyle\lim_{n\to\infty} \int_{[0,1]} \frac{nx^{n-1}}{1+x}\, d\lambda(x) = \frac{1}{2}$.

2. Let $(X, \mathcal{A}, \mu)$ be a measure space and $f \in \mathcal{L}^1(\mu) \cap \mathcal{L}^2(\mu)$.

   (a) Show that $f \in \mathcal{L}^p(\mu)$ for all $1 \leq p \leq 2$.

   (b) Prove that $\displaystyle\lim_{p \searrow 1} ||f||_p^p = ||f||_1$.