Measure and Integration Exercise 6, 2016-17

1. Let $E = \{(x,y) : 0 < x < \infty, 0 < y < 1\}$. We consider on E the restriction of the product Borel σ-algebra, and the restriction of the product Lebesgue measure $\lambda \times \lambda$. Let $f : E \to \mathbb{R}$ be given by $f(x,y) = y \sin x \, e^{-xy}$.

   (a) Show that f is $\lambda \times \lambda$ integrable on E. (2 pts)

   (b) Applying Fubini's Theorem to the function f, show that

   $$\int_0^\infty \frac{\sin x}{x} \left(\frac{1 - e^{-x}}{x} - e^{-x} \right) d\lambda(x) = \frac{1}{2} \log 2.$$

   (4pts)

2. Let $(X, \mathcal{A}, \mu)$ be a finite measure space (i.e. $\mu(X) < \infty$), and let $(u_n)_n \subset \mathcal{M}(\mathcal{A})$ be a uniformly bounded sequence (i.e. ($|u_n(x)| \leq R$ for all n and x, and some positive real number R) converging to a bounded function u in μ measure. Show that (u_n) converges to u in $\mathcal{L}^p(\mu)$ for any $p \geq 1$. (4 pts.)