Extra Practice Final Measure and Integration 2014-15

(1) Consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$ and let $f \in \mathcal{L}^1(\lambda)$ is bounded. Show that the function g defined by
$$g(t) = \int_{\mathbb{R}} \frac{f(x)}{x^2 + t^2}\, d\lambda(x),$$
is bounded and continuous on the interval $(1, \infty)$

(2) Consider the measure space $([0,1]\mathcal{B}([0,1]), \lambda)$, where $\mathcal{B}([0,1])$ is the restriction of the Borel σ-algebra to $[0,1]$, and λ is the restriction of Lebesgue measure to $[0,1]$. Let $E_1, \cdots, E_m$ be a collection of Borel measurable subsets of $[0,1]$ such that every element $x \in [0,1]$ belongs to at least n sets in the collection $\{E_j\}_{j=1}^m$, where $n \leq m$. Show that there exists a $j \in \{1, \cdots, m\}$ such that $\lambda(E_j) \geq \dfrac{n}{m}$.

(3) Let $(X, \mathcal{F}, \mu)$ be a measure space, and $1 < p, q < \infty$ conjugate numbers, i.e. $1/p + 1/q = 1$. Show that if $f \in \mathcal{L}^p(\mu)$, then there exists $g \in \mathcal{L}^q(\mu)$ such that $||g||_q = 1$ and $\int fg\, d\mu = ||f||_p$.

(4) Consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$, where $\mathcal{B}(\mathbb{R}))$ is the Borel σ-algebra and λ is Lebesgue measure. Let $f \in \mathcal{L}^1(\lambda)$ and define for $h > 0$, the function $f_h(x) = \dfrac{1}{h}\displaystyle\int_{[x,x+h]} f(t)\, d\lambda(t)$.
   (a) Show that f_h is Borel measurable for all $h > 0$.
   (b) Show that $f_h \in \mathcal{L}^1(\lambda)$ and $||f_h||_1 \leq ||f||_1$.

(5) Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space and (A_i) a sequence in $\mathcal{A}$ such that $\lim_{n\to\infty} \mu(A_n) = 0$.
   (a) Show that $\mathbf{1}_{A_n} \xrightarrow{\mu} 0$, i.e. the sequence $(\mathbf{1}_{A_n})$ converges to 0 in measure.
   (b) Show that for any $u \in \mathcal{L}^1(\mu)$, one has $u\mathbf{1}_{A_n} \xrightarrow{\mu} 0$.
   (c) Show that for any $u \in \mathcal{L}^1(\mu)$, one has
$$\sup_n \int_{\{|u|\mathbf{1}_{A_n} > |u|\}} |u|\mathbf{1}_{A_n}\, d\mu = 0.$$
   (d) Show that $\lim_{n\to\infty} \int_{A_n} u\, d\mu = 0$.

(6) Let $(X, \mathcal{A}, \mu)$ be a measure space. Show that μ is σ-finite if and only if there exists $f \in L^1(\mu)$ which is **strictly** positive.