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1. Consider a 2-period binomial model with $S_0 = 10$, $u = 1.25$, $d = 0.75$, and $r = 0.2$. Suppose the real probability measure P satisfies $P(H) = p = 0.6 = 1 - P(T)$.

   (a) Consider an option with payoff $V_2 = (\max(S_1, S_2) - 11)^+$. Determine the price V_n at time $n = 0, 1$.

   (b) Suppose $\omega_1\omega_2 = HH$, find the values of the portfolio process $\Delta_0, \Delta_1(H)$ so that the corresponding wealth process satisfies $X_2(HH) = V_2(HH)$. Describe the corresponding strategy.

   (c) Determine explicitly the state price density process

   $$\frac{Z_0}{(1+r)^0}, \frac{Z_1}{(1+r)}, \frac{Z_2}{(1+r)^2},$$

   where

   $$Z_2(\omega_1\omega_2) = Z(\omega_1\omega_2) = \frac{\widetilde{P}(\omega_1\omega_2)}{P(\omega_1\omega_2)}$$

   with $\widetilde{P}$ the risk neutral probability measure, and $Z_i = E_i(Z)$, $i = 0, 1$.

   (d) Consider the utility function $U(x) = \ln x^2$. Find a random variable X (which is a function of the two coin tosses) that maximizes $E(U(X))$ subject to the condition that $\widetilde{E}\left(\frac{X}{(1+r)^2}\right) = 10$.

2. Consider the N-period binomial model. Consider the random variables $X_1, \ldots, X_N$ on (Ω, P) defined by

   $$X_i(\omega_1 \ldots \omega_N) = \begin{cases} 2, & \text{if } \omega_i = H, \\ 0, & \text{if } \omega_i = T. \end{cases}$$

   (a) Assume $P(H) = 1/2 = P(T)$. Let $Z_0 = 1$, and $Z_n = X_1 \ldots X_n$, $n = 1, 2, \ldots, N$. Prove that the process $Z_0, Z_1, \ldots, Z_N$ is a martingale w.r.t. P.

   (b) Suppose $P(H) = 1/4 = 1 - P(T)$. Show that the process $Z_0, Z_1, \ldots, Z_N$ in part (a) is now a supermartingale w.r.t. P, while the process $Z_0^2, Z_1^2, \ldots, Z_N^2$ is a martingale w.r.t. P.

3. Consider the N-period binomial model.

(a) Assume $X_0, X_1, \ldots, X_N$ is a Markov process w.r.t. the risk neutral measure $\widetilde{P}$. Consider an option with payoff $V_N = X_N^2$. Show that for each $n = 0, 1, \ldots, N-1$, there exists a function g_n such that the price at time n is given by $V_n = g_n(X_n)$.

(b) Suppose Y is a random variable on Ω. Define a process

$$Y_0, Y_1, \ldots, Y_N$$

by $Y_n = \widetilde{E}_n(Y)$. Let

$$Z_0, Z_1, \ldots, Z_N$$

be the Radon-Nikodym derivative process of $\widetilde{P}$ w.r.t. P, so $Z_n = E_n(Z)$, with Z the Radon-Nikodym derivative of $\widetilde{P}$ w.r.t. P. Show that the process

$$Z_0Y_0, Z_1Y_1, \ldots, Y_NZ_N$$

is a martingale w.r.t. P. (Hint: use Lemma 3.2.6)