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OefenDeeltentamen 2 Inleiding Financiele Wiskunde, 2010

1. Consider a 2-period binomial model with $S_0 = 110$, $u = 1.1$, $d = .9$, and $r = 0.05$. Consider an American Put option with expiration $N = 2$ and strike price $K = 110$.

   (a) Determine the intrinsic value process G_0, G_1, G_2.

   (b) Determine the price V_n at time $n = 0, 1$ of the American put option.

   (c) Determine the optimal exercise time $\tau^*(\omega_1\omega_2)$ for all $\omega_1\omega_2$.

   (d) Suppose $\omega_1\omega_2 = HT$, find the values of the portfolio process $\Delta_0, \Delta_1(H)$ and the corresponding values of the wealth process X_0 and $X_1(H)$. Check that $X_2(HT) = V_2(HT)$.

   (e) Suppose $\omega_1\omega_2 = TT$. find $\Delta_1(T)$. Show that if the buyer exercises at time 1, then $X_1(T) = 11$, and if the buyer exercises at time 2, then $X_2(TT) = 20.9$.

2. Consider the binomial model with $u = 2^1$, $d = 2^{-1}$, and $r = 1/4$, and consider a perpetual American put option with $S_0 = 10$ and $K = 12$. Suppose that Alice and Bob each buy such an option

   (a) Suppose that Alice uses the strategy of exercising the first time the price reaches 5 euros. What should then the price be at time 0?

   (b) Suppose that Bob uses the strategy of exercising the first time the price reaches 2.5 euros. What should then the price be at time 0?

   (c) What is the probability that the price reaches 20 euros for the first time at time $n = 5$?

3. Consider the N-period Binomial model with risk neutral probability measure $\widetilde{P}$. Suppose $X_0, X_1, \cdots, X_N$ is an adapted process satisfying $X_i > -1$ for all $i = 0, 1, \cdots, N$. Define a process $Y_0, Y_1, \cdots, Y_N$ by

$$Y_0 = 1, \quad \text{and } Y_n = \frac{1}{(1 + X_0) \cdots (1 + X_{n-1})}, \; n = 1, \cdots, N.$$

   (a) Let $Z_n = \widetilde{E}_n \left[\frac{Y_N}{Y_n} \right]$, $n = 0, 1, \cdots, N$. Show that the process $Y_0Z_0, Y_1Z_1, \cdots, Y_NZ_N$ is a martingale with respect to $\widetilde{P}$.

   (b) Let $\Delta_0, \cdots, \Delta_{N-1}$ be an adapted process, and W_0 a fixed positive real number. Define for $n = 0, 1, \cdots, N - 1$,

$$W_{n+1} = \Delta_n Z_{n+1} + (1 + X_n)(W_n - \Delta_n Z_n).$$

Show that the process

$$Y_0W_0, Y_1W_1, \cdots, Y_NW_N$$

is a martingale with respect to $\widetilde{P}$.

4. Let $M_0, M_1, \cdots$ be the symmetric random walk. Define for $a \in \mathbb{Z}$, $M_n^a = a + M_n$. The process $M_0^a, M_1^a, \cdots$ is called the symmetric random walk starting in a. Let $b \in \mathbb{Z}$ be such that $n + b - a$ is even.

   (a) Let $N_n(a, b)$ be the number of paths of length n starting in a and ending in b. Show that $N_n(a, b) = \binom{n}{\frac{1}{2}(n + b - a)}$. Conclude that

   $$P(M_n^a = b) = \binom{n}{\frac{1}{2}(n + b - a)} \frac{1}{2^n}.$$

   (b) Let $N_n^0(a, b)$ be the number of paths of length n starting in a and ending in b that cross the x-axis at least once. Use the reflection principle to prove that if $a, b > 0$, then $N_n^0(a, b) = N_n(-a, b)$.

   (c) Let $b, n > 0$ be two integers satisfying $n + b$ is even. Using part (b) show that the number of paths of length n starting in 0 which does not cross the x-axis (except at the starting point) equals $\frac{b}{n} N_n(0, b)$.

   (d) Use part (c) to prove that if $b > 0$, then

   $$P(M_n = b, \min_{1 \le k \le n-1} M_k > 0) = \frac{b}{n} P(M_n = b).$$