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Uitwerkingen Oefen Deeltentamen 2 Inleiding Financiele Wiskunde, 2011-12

1. Consider a 2-period binomial model with $S_0 = 100$, $u = 1.2$, $d = 0.7$, and $r = 0.1$. Consider now an Asian American put option with expiration $N = 2$, and intrinsic value $G_n = 95 - \dfrac{S_0 + \cdots + S_n}{n+1}$, $n = 0, 1, 2$.

   (a) Determine the price V_n at time $n = 0, 1$ of this American option.

   (b) Find the optimal exercise time $\tau^*(\omega_1\omega_2)$ for all $\omega_1\omega_2$.

   (c) Suppose it is possible to buy this option at a price $C > V_0$, where V_0 is your answer from part (a). Construct an explicit arbitrage strategy.

2. Consider the binomial model with up factor $u = 2$, down factor $d = 1/2$ and interest rate $r = 1/4$. Consider a perpetual American put option with $S_0 = 8$ and strike price $K = 10$.

   (a) Suppose the buyer of the option uses the strategy of exercising the first time the price drops to 1 euro. What is then the price at time 0 of such an option?

   (b) What is the probability that the price reaches 16 euros for the first time at time $n = 5$?

3. Consider an American option with expiration date N, intrinsic value process $G_0, G_1, \cdots, G_N$, and price process $V_0, V_1, \cdots, V_N$. Note that

$$V_n = \max_{\tau \in \mathcal{S}_n} \widetilde{E}_n \left[\mathbf{1}_{\{\tau \leq N\}} \frac{G_\tau}{(1+r)^{\tau - n}} \right],$$

   for $n = 0, 1, \cdots, N$, where r is the interest rate.

   (a) For $n = 0, 1, \cdots, N$, let $\tau_n^* \in \mathcal{S}_n$ be given by $\tau_n^* = \inf\{k \geq n : V_k = G_k\}$, if the infimum exists, otherwise $\tau_n^* = \infty$. Prove that

$$\{\frac{V_{m \wedge \tau_n^*}}{(1+r)^{m \wedge \tau_n^*}}, \; m = n, \cdots, N\}$$

   is a martingale.

   (b) Use part (a) to show τ_n^* is an optimal stopping time for V_n. i.e.

$$V_n = \widetilde{E}_n \left[\mathbf{1}_{\{\tau_n^* \leq N\}} \frac{G_{\tau_n^*}}{(1+r)^{\tau_n^* - n}} \right].$$

4. Consider a 3-period (non constant interest rate) binomial model with interest rate process R_0, R_1, R_2 defined by

$$R_0 = 0,\ R_1(\omega_1) = 0.02f(\omega_1),\ R_2(\omega_1, \omega_2) = 0.02f(\omega_1)f(\omega_2)$$

where $f(H) = 3$, and $f(T) = 2$. Suppose that the risk neutral measure is given by $\widetilde{P}(HHH) = \widetilde{P}(HTT) = 1/10$, $\widetilde{P}(HHT) = \widetilde{P}(HTH) = 1/5$, $\widetilde{P}(THH) = \widetilde{P}(THT) = 1/15$, $\widetilde{P}(TTH) = \widetilde{P}(TTT) = 2/15$.

(a) Calculate the time one price $B_{1,3}$ of a zero coupon bond with maturity $m = 3$.

(b) Consider a 3-period interest rate swap. Find the 3-period swap rate SR_3, i.e. the value of K that makes the time zero no arbitrage price of the swap equal to zero.

(c) Consider a 3-period Cap that makes payments $C_n = (R_{n-1} - 0.1)^+$ at time $n = 1, 2, 3$. Find Cap_3, the price of this Cap.

5. Let $M_0, M_1, \cdots,$ be the symmetric random walk, i.e. $M_0 = 0$, and $M_n = \sum_{i=1}^n X_i$, where

$$X_i = \begin{cases} 1, & \text{if } \omega_i = H, \\ -1, & \text{if } \omega_i = T, \end{cases}$$

for $i \geq 1$. Let $m \geq 2$ be an integer, and let $k \in \{1, \cdots, m-1\}$. Define $Y_0 = k$, and

$$Y_{n+1} = (Y_n + X_{n+1})\mathbb{I}_{\{Y_n \notin \{0,m\}\}} + Y_n \mathbb{I}_{\{Y_n \in \{0,m\}\}},$$

for $n \geq 0$.

(a) Show that $Y_0, Y_1, \cdots$ is a martingale.

(b) Let $T = \inf\{n \geq 1 : Y_n \in \{0, m\}\}$. Using the the Optional Sampling Theorem show that $E(Y_T) = E(Y_0) = k$.

(c) Prove that $P(Y_T = 0) = \dfrac{m-k}{m}$.