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Practice Final Measure and Integration 2012-13

1. Let μ and ν be two measures on the measure space $(E, \mathcal{B})$ such that $\mu(A) \leq \nu(A)$ for all $A \in \mathcal{B}$.

   (a) Show that if f is any non-negative measurable function on $(E, \mathcal{B})$, then $\int_E f\, d\mu \leq \int_E f\, d\nu$.

   (b) Prove that if ν is a finite measure, then $\mathcal{L}^2(\nu) \subseteq \mathcal{L}^1(\mu)$.

2. Let $0 < a < b$. Prove with the help of Tonelli's theorem (applied to the function $f(x,y) = e^{-xt}$) that $\int_{[0,\infty)} (e^{-at} - e^{-bt}) \frac{1}{t} d\lambda(t) = \log(b/a)$, where λ denotes Lebesgue measure.

3. Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space, and (f_j) a uniformly integrable sequence of measurable functions. Define $F_k = \sup_{1 \leq j \leq k} |f_j|$ for $k \geq 1$.

   (a) Show that for any $w \in \mathcal{M}^+(\mathcal{A})$,
   $$\int_{\{F_k > w\}} F_k\, d\mu \leq \sum_{j=1}^{k} \int_{\{|f_j| > w\}} |f_j|\, d\mu.$$

   (b) Show that for every $\epsilon > 0$, there exists a $w_\epsilon \in \mathcal{L}^1_+(\mu)$ such that for all $k \geq 1$
   $$\int_X F_k\, d\mu \leq \int_X w_\epsilon\, d\mu + k\epsilon.$$

   (c) Show that
   $$\lim_{k\to\infty} \frac{1}{k} \int_X F_k\, d\mu = 0.$$

4. Consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$, where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra, and λ Lebesgue measure.

   (a) Let $f \in \mathcal{L}^1(\lambda)$. Show that for all $a \in \mathbb{R}$, one has
   $$\int_{\mathbb{R}} f(x-a) d\lambda(x) = \int_{\mathbb{R}} f(x) d\lambda(x).$$

   (b) Let $k, g \in \mathcal{L}^1(\lambda)$. Define $F : \mathbb{R}^2 \to \mathbb{R}$, and $h : \mathbb{R} \to \overline{\mathbb{R}}$ by
   $$F(x,y) = k(x-y)g(y).$$

(i) Show that F is measurable.

(ii) Show that $F \in \mathcal{L}^1(\lambda \times \lambda)$, and

$$\int_{\mathbb{R}\times\mathbb{R}} F(x,y)d(\lambda \times \lambda)(x,y) = \left(\int_{\mathbb{R}} k(x)d\lambda(x)\right)\left(\int_{\mathbb{R}} g(y)d\lambda(y)\right).$$

5. Let $(X, \mathcal{A})$ be a measurable space, and μ, ν two finite measures on $(X, \mathcal{A})$ with the property that for any $A \in \mathcal{A}$ with $\mu(A) = 0$ one has $\nu(A) = 0$. Show that for any $\epsilon > 0$ there is a $\delta > 0$ such that if $A \in \mathcal{A}$ with $\mu(A) < \delta$, then $\nu(A) < \epsilon$. (Hint: give a proof by contradiction)