Measure and Integration: Solutions Practice Final II 2015-16

(1) Consider a measure space $(X, \mathcal{A}, \mu)$, and let $(f_n)_n$ be a sequence in $\mathcal{L}^2(\mu)$ which is bounded in the $\mathcal{L}^2$ norm, i.e. there exists a constant $C > 0$ such that $||f_n||_2 < C$ for all $n \geq 1$.

(a) Prove that $\sum_{n=1}^{\infty} (\frac{f_n}{n})^2 \in \mathcal{L}^1_{\mathbb{R}}(\mu)$.

(b) Prove that $\lim_{n\to\infty} \frac{f_n}{n} = 0$ μ a.e.

(2) Consider the measure space $((0, \infty), \mathcal{B}((0, \infty)), \lambda)$, where $\mathcal{B}((0, \infty))$ is the restriction of the Borel σ-algebra, and λ Lebesgue measure restricted to $(0, \infty)$. Determine the value of

$$\lim_{n\to\infty} \int_{(0,n)} \frac{\cos(x^5)}{1 + nx^2}\, d\lambda(x).$$

(3) Let $(X, \mathcal{A}, \mu)$ be a finite measure space. Suppose $f_n, g_n, f, g \in \mathcal{M}(\mathcal{A})$ $(n \geq 1)$ satify the following:

(i) $f_n \xrightarrow{\mu} f$,

(ii) $g_n \xrightarrow{\mu} g$,

(iii) $|f_n| \leq C$ for all n, where $C > 0$.

Prove that $f_n g_n \xrightarrow{\mu} fg$.

(4) Let $(X, \mathcal{A})$ be a measurable space and μ, ν are finite measure on $\mathcal{A}$. Show that there exists a function $f \in \mathcal{L}^1_+(\mu) \cap \mathcal{L}^1_+(\nu)$ such that for every $A \in \mathcal{A}$, we have

$$\int_A (1 - f)\, d\mu = \int_A f\, d\nu.$$

(5) Let $0 < a < b$. Prove with the help of Tonelli's theorem (applied to the function $f(x, t) = e^{-xt}$) that $\int_{[0,\infty)} (e^{-at} - e^{-bt}) \frac{1}{t} d\lambda(t) = \log(b/a)$, where λ denotes Lebesgue measure.

(6) Let $(X, \mathcal{A}, \mu)$ be a finite measure space and $f_n, f \in \mathcal{M}(\mathcal{A})$, $n \geq 1$. Show that f_n converges to f in μ measure **if and only if** $\lim_{n\to\infty} \int \frac{|f_n - f|}{1 + |f_n - f|}\, d\mu = 0$.