Practice Final Measure and Integration 2014-15

(1) Let μ and ν be two measures on the measure space $(E, \mathcal{B})$ such that $\mu(A) \leq \nu(A)$ for all $A \in \mathcal{B}$.
(a) Show that if f is any non-negative measurable function on $(E, \mathcal{B})$, then $\int_E f\,d\mu \leq \int_E f\,d\nu$.
(b) Prove that if ν is a finite measure, then $\mathcal{L}^2(\nu) \subseteq \mathcal{L}^1(\mu)$.

(2) Consider the measure space $((0,1], \mathcal{B}((0,1]), \lambda)$, where $\mathcal{B}((0,1])$ and λ are the restrictions of the Borel σ-algebra and Lebesgue measure to the interval $(0,1]$. Determine the value of

$$\lim_{n\to\infty} \int_{(0,1]} e^{1/x}(1+n^2x)^{-1} \sin(ne^{-1/x}\,d\lambda(x).$$

(3) Let $(X, \mathcal{F}, \mu)$ be a **finite** measure space. Assume $f \in \mathcal{L}^2(\mu)$ satisfies $0 < ||f||_2 < \infty$, and let $A = \{x \in X : f(x) \neq 0\}$. Show that

$$\mu(A) \geq \frac{(\int f\,d\mu)^2}{\int f^2\,d\mu}.$$

bigskip

(4) Let $1 \leq p < \infty$, and suppose $(X, \mathcal{A}, \mu)$ is a finite measure space. Let $(f_n)_n \in \mathcal{L}^p(\mu)$ be a sequence converging to f in μ measure.
(a) Show that

$$\int |f|^p\,d\mu \leq \liminf_{n\to\infty} \int |f_n|^p\,d\mu.$$

(b) Show that $\lim_{n\to\infty} n^p\mu(\{|f| > n\}) = 0$.

(5) Let $E = \{(x,y) : y < x < 1, \,, 0 < y < 1\}$. We consider on E the restriction of the product Borel σ-algebra, and the restriction of the product Lebesgue measure $\lambda \times \lambda$. Let $f : E \to \mathbb{R}$ be given by $f(x,y) = x^{-3/2}\,\cos(\frac{\pi y}{2x})$.
(a) Show that f is $\lambda \times \lambda$ integrable on E.
(b) Define $F : (0,1) \to \mathbb{R}$ by $F(y) = \int_{(y,1)} x^{-3/2}\,\cos(\frac{\pi y}{2x})\,d\lambda(x)$. Determine the value of

$$\int F(y)\,d\lambda(y).$$

(6) Let $(X, \mathcal{A}, \mu)$ be a probability space (i.e. $\mu(X) = 1$) and let $\{f_n\}$ be a sequence in $\mathcal{L}^1(\mu)$ such that $\int_X |f_n| d\mu = n$ for all $n \geq 1$. Let

$$A_n = \{x : |f_n(x) - \int_X f_n d\mu| \geq n^3\}.$$

(a) Show that $\mu\left(\bigcap_{m\geq 1} \bigcup_{n\geq m} A_n\right) = 0$.
(b) Use part (a) to show that for every $\epsilon > 0$ there exists $m_0 \geq 1$ such that

$$\mu\{x \in X : |f_n(x)| < n^3 + n, \text{ for all } n \geq m_0\} > 1 - \epsilon.$$

(7) Suppose μ and ν are finite measures on the measurable space $(X, \mathcal{A})$ which have the same null sets. Show that there exists a measurable function f such that $0 < f < \infty$ μ a.e. and ν a.e. and for all $A \in \mathcal{A}$ one has

$$\nu(A) = \int_A f\,d\mu \quad \text{and} \quad \mu(A) = \int_A \frac{1}{f}\,d\nu.$$

(8) Let $(X, \mathcal{A}, \mu)$ be a finite measure space and $f_n, f \in \mathcal{M}(\mathcal{A})$, $n \geq 1$. Show that f_n converges to f in μ measure **if and only if** $\lim_{n\to\infty} \int \frac{|f_n - f|}{1 + |f_n - f|}\,d\mu = 0$.