Measure and Integration: Exercise on Radon-Nikodym Theorem, 2014-15

1. Let $(E, \mathcal{B}, \nu)$ be a measure space, and $h : E \to \mathbb{R}$ a non-negative measurable function. Define a measure μ on $(E, \mathcal{B})$ by $\mu(A) = \int_A h d\nu$ for $A \in \mathcal{B}$. Show that for every non-negative measurable function $F : E \to \mathbb{R}$ one has

$$\int_E F\, d\mu = \int_E F h\, d\nu.$$

Conclude that the result is still true for $F \in \mathcal{L}^1(\mu)$ which is not necessarily non-negative.

2. Let $(X, \mathcal{B}, \nu)$ be a measure space, and suppose $X = \bigcup_{n=1}^{\infty} E_n$, where $\{E_n\}$ is a collection of pairwise disjoint measurable sets such that $\nu(E_n) < \infty$ for all $n \geq 1$. Define μ on $\mathcal{B}$ by $\mu(B) = \sum_{n=1}^{\infty} 2^{-n} \nu(B \cap E_n)/(\nu(E_n) + 1)$.

   (a) Prove that μ is a finite measure on $(X, \mathcal{B})$.

   (b) Let $B \in \mathcal{B}$. Prove that $\mu(B) = 0$ **if and only if** $\nu(B) = 0$.

   (c) Find explicitly two positive integrable functions f and g such that

$$\mu(A) = \int_A f\, d\nu \text{ and } \nu(A) = \int_A g\, d\mu,$$

   for all $A \in \mathcal{B}$.

3. Suppose μ, ν and λ are finite measures on $(X, \mathcal{B})$ such that $\mu \ll \nu$ and $\nu \ll \lambda$. Show that $\mu \ll \lambda$ and $\frac{d\mu}{d\lambda} = \frac{d\mu}{d\nu} \cdot \frac{d\nu}{d\lambda}$ λ a.e.

4. Suppose that μ_i, ν_i are finite measures on $(X, \mathcal{A})$ with $\mu_i \ll \nu_i$ for $i = 1, 2$. Let $\nu = \nu_1 \times \nu_2$ and $\mu = \mu_1 \times \mu_2$ be the corresponding product measures on $(X \times X, \mathcal{A} \otimes \mathcal{A})$.

   (a) Show that $\mu \ll \nu$.

   (b) Prove that $\frac{d\mu}{d\nu}(x, y) = \frac{d\mu_1}{d\nu_1}(x) \cdot \frac{d\mu_2}{d\nu_2}(y)$ ν a.e.