Chapter 13

The Radon-Nikodym theorem

Suppose f is non-negative and integrable with respect to μ. If we define ν by

$$\nu(A) = \int_A f\, d\mu, \tag{13.1}$$

then ν is a measure. The only part that needs thought is the countable additivity. If A_n are disjoint measurable sets, we have

$$\nu(\cup_n A_n) = \int_{\cup_n A_n} f\, d\mu = \sum_{n=1}^\infty \int_{A_n} f\, d\mu = \sum_{n=1}^\infty \nu(A_n)$$

by using Proposition 7.5. Moreover, $\nu(A)$ is zero whenever $\mu(A)$ is.

In this chapter we consider the converse. If we are given two measures μ and ν, when does there exist f such that (13.1) holds? The Radon-Nikodym theorem answers this question.

13.1 Absolute continuity

Definition 13.1 A measure ν is said to be *absolutely continuous* with respect to a measure μ if $\nu(A) = 0$ whenever $\mu(A) = 0$. We write $\nu \ll \mu$.

Proposition 13.2 *Let ν be a finite measure. Then ν is absolutely continuous with respect to μ if and only if for all ε there exists δ such that $\mu(A) < \delta$ implies $\nu(A) < \varepsilon$.*

Proof. Suppose for each ε, there exists δ such that $\mu(A) < \delta$ implies $\nu(A) < \varepsilon$. If $\mu(A) = 0$, then $\nu(A) < \varepsilon$ for all ε, hence $\nu(A) = 0$, and thus $\nu \ll \mu$.

Suppose now that $\nu \ll \mu$. If there exists an ε for which no corresponding δ exists, then there exists E_k such that $\mu(E_k) < 2^{-k}$ but $\nu(E_k) \geq \varepsilon$. Let $F = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} E_k$. Then

$$\mu(F) = \lim_{n\to\infty} \mu(\cup_{k=n}^{\infty} E_k) \leq \lim_{n\to\infty} \sum_{k=n}^{\infty} 2^{-k} = 0,$$

but

$$\nu(F) = \lim_{n\to\infty} \nu(\cup_{k=n}^{\infty} E_k) \geq \varepsilon;$$

This contradicts the absolute continuity. □

13.2 The main theorem

Lemma 13.3 *Let μ and ν be finite positive measures on a measurable space $(X, \mathcal{A})$. Either $\mu \perp \nu$ or else there exists $\varepsilon > 0$ and $G \in \mathcal{A}$ such that $\mu(G) > 0$ and G is a positive set for $\nu - \varepsilon\mu$.*

Proof. Consider the Hahn decomposition for $\nu - \frac{1}{n}\mu$. Thus there exists a negative set E_n and a positive set F_n for this measure, E_n and F_n are disjoint, and their union is X. Let $F = \cup_n F_n$ and $E = \cap_n E_n$. Note $E^c = \cup_n E_n^c = \cup_n F_n = F$.

For each n, $E \subset E_n$, so

$$\nu(E) \leq \nu(E_n) \leq \tfrac{1}{n}\mu(E_n) \leq \tfrac{1}{n}\mu(X).$$

Since ν is a positive measure, this implies $\nu(E) = 0$.

One possibility is that $\mu(E^c) = 0$, in which case $\mu \perp \nu$. The other possibility is that $\mu(E^c) > 0$. In this case, $\mu(F_n) > 0$ for some n. Let $\varepsilon = 1/n$ and $G = F_n$. Then from the definition of F_n, G is a positive set for $\nu - \varepsilon\mu$. □

We now are ready for the *Radon-Nikodym theorem.*

Theorem 13.4 *Suppose μ is a σ-finite positive measure on a measurable space $(X, \mathcal{A})$ and ν is a finite positive measure on $(X, \mathcal{A})$ such that ν is absolutely continuous with respect to μ. Then there exists a μ-integrable non-negative function f which is measurable with respect to $\mathcal{A}$ such that*

$$\nu(A) = \int_A f \, d\mu$$

for all $A \in \mathcal{A}$. Moreover, if g is another such function, then $f = g$ almost everywhere with respect to μ.

The function f is called the *Radon-Nikodym derivative* of ν with respect to μ or sometimes the *density* of ν with respect to μ, and is written $f = d\nu/d\mu$. Sometimes one writes

$$d\nu = f \, d\mu.$$

The idea of the proof is to look at the set of f such that $\int_A f \, d\mu \leq \nu(A)$ for each $A \in \mathcal{A}$, and then to choose the one such that $\int_X f \, d\mu$ is largest.

Proof. *Step 1.* Let us first prove the uniqueness assertion. For every set A we have

$$\int_A (f - g) \, d\mu = \nu(A) - \nu(A) = 0.$$

By Proposition 8.1 we have $f - g = 0$ a.e. with respect to μ.

Step 2. Let us assume μ is a finite measure for now. In this step we define the function f. Define

$$\mathcal{F} = \Big\{ g \text{ measurable} : g \geq 0, \int_A g \, d\mu \leq \nu(A) \text{ for all } A \in \mathcal{A} \Big\}.$$

$\mathcal{F}$ is not empty because $0 \in \mathcal{F}$. Let $L = \sup\{\int g \, d\mu : g \in \mathcal{F}\}$, and let g_n be a sequence in $\mathcal{F}$ such that $\int g_n \, d\mu \to L$. Let $h_n = \max(g_1, \ldots, g_n)$.

We claim that if g_1 and g_2 are in $\mathcal{F}$, then $h_2 = \max(g_1, g_2)$ is also in $\mathcal{F}$. To see this, let $B = \{x : g_1(x) \geq g_2(x)\}$, and write

$$\begin{aligned}\int_A h_2\,d\mu &= \int_{A\cap B} h_2\,d\mu + \int_{A\cap B^c} h_2\,d\mu\\ &= \int_{A\cap B} g_1\,d\mu + \int_{A\cap B^c} g_2\,d\mu\\ &\leq \nu(A\cap B) + \nu(A\cap B^c)\\ &= \nu(A).\end{aligned}$$

Therefore $h_2 \in \mathcal{F}$.

By an induction argument, h_n is in $\mathcal{F}$.

The h_n increase, say to f. By monotone convergence, $\int f\,d\mu = L$ and

$$\int_A f\,d\mu \leq \nu(A) \tag{13.2}$$

for all A.

Step 3. Next we prove that f is the desired function. Define a measure λ by

$$\lambda(A) = \nu(A) - \int_A f\,d\mu.$$

λ is a positive measure since $f \in \mathcal{F}$.

Suppose λ is not mutually singular to μ. By Lemma 13.3, there exists $\varepsilon > 0$ and G such that G is measurable, $\mu(G) > 0$, and G is a positive set for $\lambda - \varepsilon\mu$. For any $A \in \mathcal{A}$,

$$\nu(A) - \int_A f\,d\mu = \lambda(A) \geq \lambda(A\cap G) \geq \varepsilon\mu(A\cap G) = \int_A \varepsilon\chi_G\,d\mu,$$

or

$$\nu(A) \geq \int_A (f + \varepsilon\chi_G)\,d\mu.$$

Hence $f + \varepsilon\chi_G \in \mathcal{F}$. But

$$\int_X (f+\varepsilon\chi_G)\,d\mu = L + \varepsilon\mu(G) > L,$$

a contradiction to the definition of L.

Therefore $\lambda \perp \mu$. Then there must exist $H \in \mathcal{A}$ such that $\mu(H) = 0$ and $\lambda(H^c) = 0$. Since $\nu \ll \mu$, then $\nu(H) = 0$, and hence

$$\lambda(H) = \nu(H) - \int_H f\,d\mu = 0.$$

This implies $\lambda = 0$, or $\nu(A) = \int_A f\,d\mu$ for all A.

Step 4. We now suppose μ is σ-finite. There exist $F_i \uparrow X$ such that $\mu(F_i) < \infty$ for each i. Let μ_i be the restriction of μ to F_i, that is, $\mu_i(A) = \mu(A \cap F_i)$. Define ν_i, the restriction of ν to F_i, similarly. If $\mu_i(A) = 0$, then $\mu(A \cap F_i) = 0$, hence $\nu(A \cap F_i) = 0$, and thus $\nu_i(A) = 0$. Therefore $\nu_i \ll \mu_i$. If f_i is the function such that $d\nu_i = f_i\,d\mu_i$, the argument of Step 1 shows that $f_i = f_j$ on F_i if $i \le j$. Define f by $f(x) = f_i(x)$ if $x \in F_i$. Then for each $A \in \mathcal{A}$,

$$\nu(A \cap F_i) = \nu_i(A) = \int_A f_i\,d\mu_i = \int_{A\cap F_i} f\,d\mu.$$

Letting $i \to \infty$ shows that f is the desired function. □

13.3 Lebesgue decomposition theorem

The proof of the *Lebesgue decomposition theorem* is almost the same.

Theorem 13.5 *Suppose μ and ν are two finite positive measures. There exist positive measures λ, ρ such that $\nu = \lambda + \rho$, ρ is absolutely continuous with respect to μ, and λ and μ are mutually singular.*

Proof. Define $\mathcal{F}$ and L and construct f as in the proof of the Radon-Nikodym theorem. Let $\rho(A) = \int_A f\,d\mu$ and let $\lambda = \nu - \rho$. Our construction shows that

$$\int_A f\,d\mu \le \nu(A),$$

so $\lambda(A) \ge 0$ for all A. We have $\rho + \lambda = \nu$. We need to show μ and λ are mutually singular.

If not, by Lemma 13.3, there exists $\varepsilon > 0$ and $F \in \mathcal{A}$ such that $\mu(F) > 0$ and F is a positive set for $\lambda - \varepsilon\mu$. We get a contradiction exactly as in the proof of the Radon-Nikodym theorem. We conclude that $\lambda \perp \mu$. □

13.4 Exercises

Exercise 13.1 This exercise asks you to prove the Radon-Nikodym theorem for signed measures. Let $(X, \mathcal{A})$ be a measurable space. Suppose ν is a signed measure, μ is a finite positive measure, and $\nu(A) = 0$ whenever $\mu(A) = 0$ and $A \in \mathcal{A}$. Show there exists an integrable real-valued function f such that $\nu(A) = \int_A f\,d\mu$ for all $A \in \mathcal{A}$.

Exercise 13.2 We define a *complex measure* μ on a measurable space $(X, \mathcal{A})$ to be a map from $\mathcal{A}$ to $\mathbb{C}$ such that $\mu(\emptyset) = 0$ and $\mu(\cup_{i=1}^\infty A_i) = \sum_{i=1}^\infty \mu(A_i)$ whenever the A_i are in $\mathcal{A}$ and are pairwise disjoint. Formulate and prove a Radon-Nikodym theorem for complex measures.

Exercise 13.3 Let $(X, \mathcal{A})$ be a measurable space and let μ and ν be two finite measures. We say μ and ν are *equivalent measures* if $\mu \ll \nu$ and $\nu \ll \mu$. Show that μ and ν are equivalent if and only if there exists a measurable function f that is strictly positive a.e. with respect to μ such that $d\nu = f\,d\mu$.

Exercise 13.4 Suppose μ and ν are two finite measures such that ν is absolutely continuous with respect to μ. Let $\rho = \mu + \nu$. Note that $\mu(A) \le \rho(A)$ and $\nu(A) \le \rho(A)$ for each measurable A. In particular, $\mu \ll \rho$ and $\nu \ll \rho$. Prove that if $f = d\mu/d\rho$ and $g = d\nu/d\rho$, then g is strictly positive for almost every x with respect to μ, $f + g = 1$, and $d\nu = (f/g)\,d\mu$.

Exercise 13.5 If μ is a signed measure on $(X, \mathcal{A})$ and $|\mu|$ is the total variation measure, prove that there exists a real-valued function f that is measurable with respect to $\mathcal{A}$ such that $|f| = 1$ a.e. with respect to μ and $d\mu = f\,d|\mu|$.

Exercise 13.6 Suppose $\nu \ll \mu$ and $\rho \ll \nu$. Prove that $\rho \ll \mu$ and

$$\frac{d\rho}{d\mu} = \frac{d\rho}{d\nu} \cdot \frac{d\nu}{d\mu}.$$

Exercise 13.7 Suppose λ_n is a sequence of positive measures on a measurable space $(X, \mathcal{A})$ with $\sup_n \lambda_n(X) < \infty$ and μ is another finite positive measure on $(X, \mathcal{A})$. Suppose $\lambda_n = f_n\,d\mu + \nu_n$ is the Lebesgue decomposition of λ_n; in particular, $\nu_n \perp \mu$. If $\lambda = \sum_{n=1}^{\infty} \lambda_n$, show that

$$\lambda = \Big(\sum_{n=1}^{\infty} f_n\Big)\, d\mu + \sum_{n=1}^{\infty} \nu_n$$

is the Lebesgue decomposition of λ.

Exercise 13.8 Let $(X, \mathcal{F}, \mu)$ be a measure space, and suppose $\mathcal{E}$ is a sub-σ-algebra of $\mathcal{F}$, that is, $\mathcal{E}$ is itself a σ-algebra and $\mathcal{E} \subset \mathcal{F}$. Suppose f is a non-negative integrable function that is measurable with respect to $\mathcal{F}$. Define $\nu(A) = \int_A f\, d\mu$ for $A \in \mathcal{E}$ and let $\overline{\mu}$ be the restriction of μ to $\mathcal{E}$.
(1) Prove that $\nu \ll \overline{\mu}$.
(2) Since ν and $\overline{\mu}$ are measures on $\mathcal{E}$, then $g = d\nu/d\overline{\mu}$ is measurable with respect to $\mathcal{E}$. Prove that

$$\int_A g\, d\mu = \int_A f\, d\mu \qquad (13.3)$$

whenever $A \in \mathcal{E}$. g is called the *conditional expectation* of f with respect to $\mathcal{E}$ and we write $g = \mathbb{E}\,[f \mid \mathcal{E}]$. If f is integrable and real-valued but not necessarily non-negative, we define

$$\mathbb{E}\,[f \mid \mathcal{E}] = \mathbb{E}\,[f^+ \mid \mathcal{E}] - \mathbb{E}\,[f^- \mid \mathcal{E}].$$

(3) Show that $f = g$ if and only if f is measurable with respect to $\mathcal{E}$.
(4) Prove that if h is $\mathcal{E}$ measurable and $\int_A h\, d\mu = \int_A f\, d\mu$ for all $A \in \mathcal{E}$, then $h = g$ a.e. with respect to μ.

Exercise 13.9 Suppose $(X, \mathcal{A}, \mu)$ is a measure space and f is integrable and measurable with respect to $\mathcal{A}$. Suppose in addition that $B_1, B_2, \ldots, B_n$ is a finite sequence of disjoint elements of $\mathcal{A}$ whose union is X and that each B_j has positive μ measure. Let $\mathcal{C} = \sigma(B_1, \ldots, B_n)$. Prove that

$$\mathbb{E}\,[f \mid \mathcal{C}] = \sum_{j=1}^{n} \frac{\int_{B_j} f\, d\mu}{\mu(B_j)} \chi_{B_j}.$$

Exercise 13.10 Suppose that $(X, \mathcal{F}, \mu)$ is a measure space, $\mathcal{E}$ is a sub-σ-algebra of $\mathcal{F}$, and $\mathcal{D}$ is a sub-σ-algebra of $\mathcal{E}$. Suppose f is integrable, real-valued, and measurable with respect to $\mathcal{F}$. Prove that

$$\mathbb{E}\,[\mathbb{E}\,[f \mid \mathcal{E}] \mid \mathcal{D}] = \mathbb{E}\,[f \mid \mathcal{D}]$$

and

$$\mathbb{E}\,[\mathbb{E}\,[f \mid \mathcal{D}] \mid \mathcal{E}] = \mathbb{E}\,[f \mid \mathcal{D}].$$

Exercise 13.11 Suppose that $(X, \mathcal{F}, \mu)$ is a measure space and $\mathcal{E}$ is a sub-σ-algebra of $\mathcal{F}$. Suppose that f and fg are integrable real-valued functions, where f is measurable with respect to $\mathcal{F}$ and g is measurable with respect to $\mathcal{E}$. Prove that

$$\mathbb{E}\,[fg \mid \mathcal{E}] = g\mathbb{E}\,[f \mid \mathcal{E}].$$

Chapter 14

Differentiation

In this chapter we want to look at when a function from $\mathbb{R}$ to $\mathbb{R}$ is differentiable and when the fundamental theorem of calculus holds. Briefly, our results are the following.
(1) The derivative of $\int_a^x f(y)\,dy$ is equal to f a.e. if f is integrable (Theorem 14.5);
(2) Functions of bounded variation, in particular monotone functions, are differentiable (Theorem 14.8);
(3) $\int_a^b f'(y)\,dy = f(b) - f(a)$ if f is absolutely continuous (Theorem 14.14).

Our approach uses what are known as maximal functions and uses the Radon-Nikodym theorem and the Lebesgue decomposition theorem. However, some students and instructors prefer a more elementary proof of the results on differentiation. In Sections 14.5, 14.6, and 14.7 we give an alternative approach that avoids the use of the Radon-Nikodym theorem and Lebesgue decomposition theorem.

The definition of derivative is the same as in elementary calculus. A function f is *differentiable* at x if

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

exists, and the limit is called the *derivative* of f at x and is denoted $f'(x)$. If $f : [a,b] \to \mathbb{R}$, we say f is differentiable on $[a,b]$ if the