Chapter 12

Signed measures

Signed measures have the countable additivity property of measures, but are allowed to take negative as well as positive values. We will see shortly that an example of a signed measure is $\nu(A) = \int_A f\,d\mu$, where f is integrable and takes both positive and negative values.

12.1 Positive and negative sets

Definition 12.1 Let $\mathcal{A}$ be a σ-algebra. A *signed measure* is a function $\mu : \mathcal{A} \to (-\infty, \infty]$ such that $\mu(\emptyset) = 0$ and $\mu(\cup_{i=1}^\infty A_i) = \sum_{i=1}^\infty \mu(A_i)$ whenever the A_i are pairwise disjoint and all the A_i are in $\mathcal{A}$.

When we want to emphasize that a measure is defined as in Definition 3.1 and only takes non-negative values, we refer to it as a *positive measure*.

Definition 12.2 Let μ be a signed measure. A set $A \in \mathcal{A}$ is called a *positive set* for μ if $\mu(B) \geq 0$ whenever $B \subset A$ and $B \in \mathcal{A}$. We say $A \in \mathcal{A}$ is a *negative set* if $\mu(B) \leq 0$ whenever $B \subset A$ and $B \in \mathcal{A}$. A *null set* A is one where $\mu(B) = 0$ whenever $B \subset A$ and $B \in \mathcal{A}$.

Note that if μ is a signed measure, then

$$\mu(\cup_{i=1}^{\infty} A_i) = \lim_{n\to\infty} \mu(\cup_{i=1}^{n} A_i).$$

The proof is the same as in the case of positive measures.

Example 12.3 Suppose m is Lebesgue measure and

$$\mu(A) = \int_A f\,dm$$

for some integrable f. If we let $P = \{x : f(x) \geq 0\}$, then P is easily seen to be a positive set, and if $N = \{x : f(x) < 0\}$, then N is a negative set. The Hahn decomposition which we give below is a decomposition of our space (in this case $\mathbb{R}$) into the positive and negative sets P and N. This decomposition is unique, except that $C = \{x : f(x) = 0\}$ could be included in N instead of P, or apportioned partially to P and partially to N. Note, however, that C is a null set. The Jordan decomposition below is a decomposition of μ into μ^+ and μ^-, where $\mu^+(A) = \int_A f^+\,dm$ and $\mu^-(A) = \int_A f^-\,dm$.

Proposition 12.4 *Let μ be a signed measure which takes values in $(-\infty, \infty]$. Let E be measurable with $\mu(E) < 0$. Then there exists a measurable subset F of E that is a negative set with $\mu(F) < 0$.*

Proof. If E is a negative set, we are done. If not, there exists a measurable subset with positive measure. Let n_1 be the smallest positive integer such that there exists $E_1 \subset E$ with $\mu(E_1) \geq 1/n_1$. We then define pairwise disjoint measurable sets $E_2, E_3, \ldots$ by induction as follows. Let $k \geq 2$ and suppose $E_1, \ldots, E_{k-1}$ are pairwise disjoint measurable sets with $\mu(E_i) > 0$ for $i = 1, \ldots, k-1$. If $F_k = E - (E_1 \cup \cdots \cup E_{k-1})$ is a negative set, then

$$\mu(F_k) = \mu(E) - \sum_{i=1}^{k-1} \mu(E_i) \leq \mu(E) < 0$$

and F_k is the desired set F. If F_k is not a negative set, let n_k be the smallest positive integer such that there exists $E_k \subset F_k$ with E_k measurable and $\mu(E_k) \geq 1/n_k$.

We stop the construction if there exists k such that F_k is a negative set with $\mu(F_k) < 0$. If not, we continue and let $F = \cap_k F_k = E - (\cup_k E_k)$. Since $0 > \mu(E) > -\infty$ and $\mu(E_k) \geq 0$, then

$$\mu(E) = \mu(F) + \sum_{k=1}^{\infty} \mu(E_k).$$

Then $\mu(F) \leq \mu(E) < 0$, so the sum converges.

It remains to show that F is a negative set. Suppose $G \subset F$ is measurable with $\mu(G) > 0$. Then $\mu(G) \geq 1/N$ for some N. But this contradicts the construction, since for some k, $n_k > N$, and we would have chosen the set G instead of the set E_k at stage k. Therefore F must be a negative set. $\square$

12.2 Hahn decomposition theorem

Recall that we write $A \Delta B$ for $(A - B) \cup (B - A)$. The following is known as the *Hahn decomposition*.

Theorem 12.5 *(1) Let μ be a signed measure taking values in $(-\infty, \infty]$. There exist disjoint measurable sets E and F in $\mathcal{A}$ whose union is X and such that E is a negative set and F is a positive set.*

(2) If E' and F' are another such pair, then $E \Delta E' = F \Delta F'$ is a null set with respect to μ.

(3) If μ is not a positive measure, then $\mu(E) < 0$. If $-\mu$ is not a positive measure, then $\mu(F) > 0$.

Proof. (1) Let $L = \inf\{\mu(A) : A \text{ is a negative set}\}$. Choose negative sets A_n such that $\mu(A_n) \to L$. Let $E = \cup_{n=1}^{\infty} A_n$. Let $B_n = A_n - (B_1 \cup \cdots \cup B_{n-1})$ for each n. Since A_n is a negative set, so is each B_n. Also, the B_n are disjoint and $\cup_n B_n = \cup_n A_n = E$. If $C \subset E$, then

$$\mu(C) = \lim_{n \to \infty} \mu(C \cap (\cup_{i=1}^{n} B_i)) = \lim_{n \to \infty} \sum_{i=1}^{n} \mu(C \cap B_i) \leq 0.$$

Thus E is a negative set.

Since E is a negative set,

$$\mu(E) = \mu(A_n) + \mu(E - A_n) \le \mu(A_n).$$

Letting $n \to \infty$, we obtain $\mu(E) = L$.

Let $F = E^c$. If F were not a positive set, there would exist $B \subset F$ with $\mu(B) < 0$. By Proposition 12.4 there exists a negative set C contained in B with $\mu(C) < 0$. But then $E \cup C$ would be a negative set with $\mu(E \cup C) < \mu(E) = L$, a contradiction.

(2) To prove uniqueness, if E', F' are another such pair of sets and $A \subset E - E' \subset E$, then $\mu(A) \le 0$. But $A \subset E - E' = F' - F \subset F'$, so $\mu(A) \ge 0$. Therefore $\mu(A) = 0$. The same argument works if $A \subset E' - E$, and any subset of $E \Delta E'$ can be written as the union of A_1 and A_2, where $A_1 \subset E - E'$ and $A_2 \subset E' - E$.

(3) Suppose μ is not a positive measure but $\mu(E) = 0$. If $A \in \mathcal{A}$, then

$$\mu(A) = \mu(A \cap E) + \mu(A \cap F) \ge \mu(E) + \mu(A \cap F) \ge 0,$$

which says that μ must be a positive measure, a contradiction. A similar argument applies for $-\mu$ and F. $\square$

Let us say two measures μ and ν are *mutually singular* if there exist two disjoint sets E and F in $\mathcal{A}$ whose union is X with $\mu(E) = \nu(F) = 0$. This is often written $\mu \perp \nu$.

Example 12.6 If μ is Lebesgue measure restricted to $[0, 1/2]$, that is, $\mu(A) = m(A \cap [0, 1/2])$, and ν is Lebesgue measure restricted to $[1/2, 1]$, then μ and ν are mutually singular. We let $E = [0, 1/2]$ and $F = (1/2, 1]$. This example works because the Lebesgue measure of $\{1/2\}$ is 0.

Example 12.7 A more interesting example is the following. Let f be the Cantor-Lebesgue function where we define $f(x) = 1$ if $x \ge 1$ and $f(x) = 0$ if $x \le 0$ and let ν be the Lebesgue-Stieltjes measure associated with f. Let μ be Lebesgue measure. Then $\mu \perp \nu$. To see this, we let $E = C$, where C is the Cantor set, and $F = C^c$. We already know that $m(E) = 0$ and we need to show $\nu(F) = 0$. To do that, we need to show $\nu(I) = 0$ for every open interval contained in F. This will follow if we show $\nu(J) = 0$ for every interval of the form $J = (a, b]$ contained in F. But f is constant on every such interval, so $f(b) = f(a)$, and therefore $\nu(J) = f(b) - f(a) = 0$.

12.3 Jordan decomposition theorem

The following is known as the *Jordan decomposition theorem.*

Theorem 12.8 *If μ is a signed measure on a measurable space $(X, \mathcal{A})$, there exist positive measures μ^+ and μ^- such that $\mu = \mu^+ - \mu^-$ and μ^+ and μ^- are mutually singular. This decomposition is unique.*

Proof. Let E and F be negative and positive sets, resp., for μ so that $X = E \cup F$ and $E \cap F = \emptyset$. Let $\mu^+(A) = \mu(A \cap F)$, $\mu^-(A) = -\mu(A \cap E)$. This gives the desired decomposition.

If $\mu = \nu^+ - \nu^-$ is another such decomposition with ν^+, ν^- mutually singular, let E' be a set such that $\nu^+(E') = 0$ and $\nu^-((E')^c) = 0$. Set $F' = (E')^c$. Hence $X = E' \cup F'$ and $E' \cap F' = \emptyset$. If $A \subset F'$, then $\nu^-(A) \le \nu^-(F') = 0$, and so

$$\mu(A) = \nu^+(A) - \nu^-(A) = \nu^+(A) \ge 0,$$

and consequently F' is a positive set for μ. Similarly, E' is a negative set for μ. Thus E', F' gives another Hahn decomposition of X. By the uniqueness part of the Hahn decomposition theorem, $F \Delta F'$ is a null set with respect to μ. Since $\nu^+(E') = 0$ and $\nu^-(F') = 0$, if $A \in \mathcal{A}$, then

$$\begin{aligned} \nu^+(A) &= \nu^+(A \cap F') = \nu^+(A \cap F') - \nu^-(A \cap F') \\ &= \mu(A \cap F') = \mu(A \cap F) = \mu^+(A), \end{aligned}$$

and similarly $\nu^- = \mu^-$. $\square$

The measure
$$|\mu| = \mu^+ + \mu^- \tag{12.1}$$
is called the *total variation measure* of μ.

12.4 Exercises

Exercise 12.1 Suppose μ is a signed measure. Prove that A is a null set with respect to μ if and only if $|\mu|(A) = 0$.

Exercise 12.2 Let μ be a signed measure. Define

$$\int f\,d\mu = \int f\,d\mu^+ - \int f\,d\mu^-.$$

Prove that

$$\Big|\int f\,d\mu\Big| \le \int |f|\,d|\mu|.$$

Exercise 12.3 Let μ be a signed measure on $(X, \mathcal{A})$. Prove that

$$|\mu(A)| = \sup\Big\{\Big|\int_A f\,d\mu\Big| : f \le 1\Big\}.$$

Exercise 12.4 Let μ be a positive measure and ν a signed measure. Prove that $\nu \ll \mu$ if and only if $\nu^+ \ll \mu$ and $\nu^- \ll \mu$.

Exercise 12.5 Let $(X, \mathcal{A})$ be a measurable space. Suppose $\lambda = \mu - \nu$, where μ and ν are finite positive measures. Prove that $\mu(A) \ge \lambda^+(A)$ and $\nu(A) \ge \lambda^-(A)$ for every $A \in \mathcal{A}$.

Exercise 12.6 Let $(X, \mathcal{A})$ be a measurable space. Prove that if μ and ν are finite signed measures, then $|\mu + \nu|(A) \le |\mu(A)| + |\nu(A)|$ for every $A \in \mathcal{A}$.

Exercise 12.7 Suppose that μ is a signed measure on $(X, \mathcal{A})$. Prove that if $A \in \mathcal{A}$, then

$$\mu^+(A) = \sup\{\mu(B) : B \in \mathcal{A}, B \subset A\}$$

and

$$\mu^-(A) = -\inf\{\mu(B) : B \in \mathcal{A}, B \subset A\}.$$

Exercise 12.8 Suppose that μ is a signed measure on $(X, \mathcal{A})$. Prove that if $A \in \mathcal{A}$, then

$$|\mu|(A) = \sup\Big\{\sum_{j=1}^n |\mu(B_j)| : \text{ each } B_j \in \mathcal{A}, \\ \text{the } B_j \text{ are disjoint}, \cup_{j=1}^n B_j = A\Big\}.$$