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Measure and Integration 2012-13-Selected Solutions Chapter 8

1. (**Exercise 8.2, p.65**) Define

$$\mathcal{B}(\overline{\mathbb{R}}) = \{B \cup C : B \in \mathcal{B}(\mathbb{R}),\ C \in \{\emptyset, \{-\infty\}, \{+\infty\}, \{-\infty, +\infty\}\}.$$

Show that $\mathcal{B}(\overline{\mathbb{R}})$ is a σ-algebra over $\overline{\mathbb{R}}$. Moreover prove that $\mathcal{B}(\mathbb{R}) = \mathbb{R} \cap \mathcal{B}(\overline{\mathbb{R}})$.

Proof: Clearly $\emptyset \in \mathcal{B}(\overline{\mathbb{R}})$ since $\emptyset = \emptyset \cup \emptyset$ and $\emptyset \in \mathcal{B}(\mathbb{R})$. Suppose $\overline{B} \in \mathcal{B}(\overline{\mathbb{R}})$, then $\overline{B} = B \cup C$ with $B \in \mathcal{B}(\mathbb{R})$ and $C \in \{\emptyset, \{-\infty\}, \{+\infty\}, \{-\infty, +\infty\}$. Now, $\overline{B}^c = \overline{\mathbb{R}} \setminus \overline{B} = (\overline{\mathbb{R}} \setminus B) \cap (\overline{\mathbb{R}} \setminus C)$. Since $B \subset \mathbb{R}$, then $\overline{\mathbb{R}} \setminus B$ contains $\{-\infty, +\infty\}$, so that $\overline{\mathbb{R}} \setminus B = (\mathbb{R} \setminus B) \cup \{-\infty, +\infty\}$. Furthermore, $\mathbb{R} \subset \overline{\mathbb{R}} \setminus C$, hence

$$\begin{aligned}
\overline{B}^c &= [(\mathbb{R} \setminus B) \cup \{-\infty, +\infty\}] \cap (\overline{\mathbb{R}} \setminus C) \\
&= [(\mathbb{R} \setminus B) \cap (\overline{\mathbb{R}} \setminus C)] \cup [\{-\infty, +\infty\} \cap (\overline{\mathbb{R}} \setminus C)] \\
&= (\mathbb{R} \setminus B) \cup [\{-\infty, +\infty\} \cap (\overline{\mathbb{R}} \setminus C)].
\end{aligned}$$

Since $\mathbb{R} \setminus B \in \mathcal{B}(\mathbb{R})$ and $\{-\infty, +\infty\} \cap (\overline{\mathbb{R}} \setminus C) \in \{\emptyset, \{-\infty\}, \{+\infty\}, \{-\infty, +\infty\}\}$, it follows that $\overline{B}^c \in \mathcal{B}(\overline{\mathbb{R}})$. Finally, let $\overline{B}_n \in \mathcal{B}(\overline{\mathbb{R}})$. Then, $\overline{B}_n = B_n \cup C_n$ with $B_n \in \mathcal{B}(\mathbb{R})$ and $C_n \in \{\emptyset, \{-\infty\}, \{+\infty\}, \{-\infty, +\infty\}\}$. Now,

$$\bigcup_n \overline{B}_n = (\bigcup_n B_n) \cup (\bigcup_n C_n) \in \mathcal{B}(\overline{\mathbb{R}}),$$

since $\bigcup_n B_n \in \mathcal{B}(\overline{\mathbb{R}})$ and $\bigcup_n C_n \in \{\emptyset, \{-\infty\}, \{+\infty\}, \{-\infty, +\infty\}\}$.

We now show that $\mathcal{B}(\mathbb{R}) = \mathcal{B}(\overline{\mathbb{R}}) \cap \mathbb{R}$. Clearly, $\mathcal{B}(\mathbb{R}) \subset \mathcal{B}(\overline{\mathbb{R}}) \cap \mathbb{R}$. Now let $D \in \mathcal{B}(\overline{\mathbb{R}}) \cap \mathbb{R}$, then $D = \overline{B} \cap \mathbb{R}$ where $\overline{B} = B \cup C$ with $B \in \mathcal{B}(\mathbb{R})$ and $C \in \{\emptyset, \{-\infty\}, \{+\infty\}, \{-\infty, +\infty\}$. Hence, $D = B \in \mathcal{B}(\mathbb{R})$. This shows that $\mathcal{B}(\overline{\mathbb{R}}) \cap \mathbb{R} \subset \mathcal{B}(\mathbb{R})$, and therefore $\mathcal{B}(\mathbb{R}) = \mathcal{B}(\overline{\mathbb{R}}) \cap \mathbb{R}$.

2. (**p. 65 exercise 8.3**) Let $(X, \mathcal{A})$ be a measurable space.

   (a) Let $f, g : X \to \mathbb{R}$ be measurable fuctions and let $A \in \mathcal{A}$. Show that the function $h : X \to R$ defined by $h(x) = f(x)$ if $x \in A$, and $h(x) = g(x)$ if $x \notin A$ is measurable.

   (b) Let $(f_j)_j$ be a sequence of measurable functions and let $(A_j)_j \subset \mathcal{A}$ be such that $X = \bigcup_j A_j$ and $f_j = f_k$ on $A_j \cap A_k$. Define $f : X \to \mathbb{R}$ by $f(x) = f_j$ if $x \in A_j$. Show that f is measurable.

Proof(a): Notice that since $A, A^c \in \mathcal{A}$, then the indicator functions 1_A and 1_{A^c} are measurable. Furthermore, $h(x) = f(x) \cdot 1_A(x) + g(x) \cdot 1_{A^c}$, hence measurable (we used the fact the sums and products of measurable functions are measurable).

Proof(b): Notice that the condition $f_j = f_k$ on $A_j \cap A_k$ implies that f is well-defined. Let $B \in \mathcal{B}(\mathbb{R})$, then

$$f^{-1}(B) = f^{-1}(B) \cap \bigcup_j A_j = \bigcup_j (f^{-1}(B) \cap A_j) = \bigcup_j (f_j^{-1}(B) \cap A_j) \in \mathcal{A}.$$

Therefore, f is meaurable.

3. (**Exercise 8.9, p.65**) Show that the functions $f, g : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \max\{x, 0\}$ and $g(x) = \min\{x, 0\}$ are continuous and hence $\mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R})$ measurable. Conclude that if $(X, \mathcal{A})$ is a measure space and $u : X \to \mathbb{R}$ is an $\mathcal{A}/\mathcal{B}(\mathbb{R})$ measurable function, then the positive part u^+ and the negative part u^- are also $\mathcal{A}/\mathcal{B}(\mathbb{R})$ measurable.

   Proof: Notice that the functions $i, k : \mathbb{R} \to \mathbb{R}$ given by $i(x) = x$ and $k(x) = |x|$ are continuous. Now, $f(x) = \frac{1}{2}(i(x) + k(x))$ and $g(x) = \frac{1}{2}(-i(x) + k(x))$ are linear combinations of continuous functions, hence continuous and therefore, $\mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R})$ measurable. Finally, if $u : X \to \mathbb{R}$ is an $\mathcal{A}/\mathcal{B}(\mathbb{R})$ measurable function, then $u^+ = f \circ u$ and $u^- = g \circ u$ are compositions of measurable functions, hence measurable.

4. (**p.65 exercise 8.12**) Let $u : \mathbb{R} \to \mathbb{R}$ be differentiable. Explain why u and $u' = du/dx$ are measurable.

   Proof: Since u is differentiable, then u is continuous and hence measurable. Define $u_n : \mathbb{R} \to \mathbb{R}$ by $u_n = n\,(u(x + 1/n) - u(x))$. Notice that u_n is a linear combination of measurable functions, hence measurable. Also $u'(x) = \lim_{n\to\infty} u_n(x)$ for all $x \in \mathbb{R}$. Therefore by Corollary 8.9, u' is measurable.

5. (**Exercise 8.15, p.65**) Let λ be the one-dimensional Lebesgue measure and $u : \mathbb{R} \to \mathbb{R}$ given by $u(x) = |x|$. Determine the measure $u(\lambda) = \lambda \circ u^{-1}$.

   Proof: Notice that $u(\mathbb{R}) = [0, \infty)$. Hence for all Borel sets $B \subset (-\infty, 0)$, one has $u(\lambda)(B) = \lambda(u^{-1}(B)) = \lambda(\emptyset) = 0$. We therefore need to determine $\lambda \circ u^{-1}$ on $\mathcal{B}(\mathbb{R}) \cap [0, \infty)$. Suppose $(a, b) \subset [0, \infty)$ is an interval, then

   $$\begin{aligned} u(\lambda)(a,b) &= \lambda(u^{-1}((a,b))) = \lambda((-b,-a) \cup (a,b)) \\ &= (-a - (-b)) + (b - a) = 2(b - a) = 2\lambda((a,b)). \end{aligned}$$

   Since $\{[0, k)\}$ is an exhaustung sequence of finite $u(\lambda)$, by Theorem 5.7, we see that $u(\lambda) = 2\lambda$.