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Measure and Integration 2012-13-Selected Solutions Chapter 7

1. (**Exercise 7.7, p.54**). Use image measures to give a new proof that $\lambda^n(t\cdot B) = t^n\lambda^n(B)$ for all $B \in \mathcal{B}(\mathbb{R}^n)$ and for all $t > 0$.

   Proof: Let $t > 0$ and A the $n \times n$ diagonal matrix such that each diagonal entry is $1/t$. Define $T_t : \mathbb{R}^n \to \mathbb{R}^n$ by $T_t(x) = Ax$, i.e. $T_t(x_1, \cdots, x_n) = (x_1/t, \cdots, x_n/t)$. Clearly T_t is continuous and hence measurable. Notice that $T_t^{-1}B = t \cdot B = AB$, hence $T_t(\lambda^n)(B) = \lambda^n(T_t^{-1}(B)) = \lambda^n(t \cdot B)$ for all $B \in \mathcal{B}(\mathbb{R}^n)$. Since T_t^{-1} is a linear transformation, we have by Theorem 7.10 that

   $$T_t(\lambda^n)(B) = |\det T_t^{-1}|\lambda^n(B) = t^n\lambda^n(B)$$

   for all $B \in \mathcal{B}(\mathbb{R}^n)$ and for all $t > 0$.

2. (**Exercise 7.8, p.54**) Let $T : X \to Y$ be any map. Show that $T^{-1}(\sigma(\mathcal{G})) = \sigma(T^{-1}(\mathcal{G})$ holds for arbitrary families $\mathcal{G}$ of subsets of Y.

   Proof: First note that $T^{-1}(\sigma(\mathcal{G}))$ is a σ-algebra (see example 3.3(vii). Since $\mathcal{G} \subset \sigma(\mathcal{G})$, then $T^{-1}\mathcal{G} \subset T^{-1}(\sigma(\mathcal{G}))$ and hence $\sigma(T^{-1}(\mathcal{G})) \subset T^{-1}(\sigma(\mathcal{G}))$ (since $\sigma(T^{-1}(\mathcal{G}))$ is the smallest σ-algebra containing $T^{-1}(\mathcal{G})$). Furthermore, $T^{-1}(\mathcal{G}) \subset \sigma(T^{-1}(\mathcal{G})$, hence by Lemma 7.2, we have $T^{-1}(\sigma(\mathcal{G})) \subset \sigma(T^{-1}(\mathcal{G})$. Thus, $T^{-1}(\sigma(\mathcal{G})) = \sigma(T^{-1}(\mathcal{G})$.

3. (**Exercise 7.10, p.55**) (Cantor's ternary set). Let $(X, \mathcal{A}) = [0, 1], [0, 1] \cap \mathcal{B}(\mathbb{R}))$, $\lambda = \lambda^1|_{[0,1]}$, and $E_0 = [0, 1]$. Remove the open middle third of E_0 to get two disjoint intervals whose union is E_1. Remove the open middle third from each of the intervals making up E_1 to get four disjoint intervals whose union is E_2, etc...

   (i) Describe explicitely E_0, E_1, E_2, E_3.

   Proof: $E_0 = [0, 1]$, $E_1 = [0, 1/3] \cup [2/3, 1]$,
   $E_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$
   $E_3 = [0, 1/27] \cup [2/27, 1/9] \cup [2/9, 7/29] \cup [8/27, 1/3] \cup [2/3, 19/27] \cup [20/27, 7/9] \cup [8/9, 25/27] \cup [26/27, 1]$.

   (ii) Prove that each E_n is compact (i.e. closed and bounded). Conclude that $C = \cup_{n\geq 0}E_n$ is non-empty and compact.

   Proof: Each E_n is a finite union of 2^n compact intervals, hence E_n is compact (note that finite union of closed and bounded sets is itself closed and bounded). Since $E_0 \supset E_1 \supset E_2 \supset \ldots$, then C is also compact (note that arbitrary intersection of closed sets is closed and since each set is bounded then C is closed and bounded).

We now prove that C is non-empty. For each n, choose a point $x_n \in E_n$ (note that $x_n \in E_i$ for $i \le n$). Then (x_n) is a sequence in $E_0 = [0,1]$. Since E_0 is compact, then (x_n) has a convergent sequence (x_{n_i}) converging to some point $x \in E_0$. Notice that for each n $x_{n_i} \in E_n$ for all n_i sufficiently large (namely for all $n_i \ge n$). Since E_n is compact, then the limit point x must lie in E_n for all n. Hence, $x \in C$ and C is non-empty.

(iii) Prove that $C \cap \bigcup_{n\in\mathbb{N}} \bigcup_{k\ge 0} (\frac{3k+1}{3^n}, \frac{3k+2}{3^n}) = \emptyset$.

Proof: Notice that at stage n, the sets removed from E_{n-1} in order to get E_n are all of the form $(\frac{3k+1}{3^n}, \frac{3k+2}{3^n})$ with $n > 0$ and $k \ge 0$. Hence, C is disjoint from any such interval and therefore disjoint from the union of such intervals.

(iv) Determine $\lambda(E_n)$ and show that $\lambda(C) = 0$.

Proof: For each n, E_n consists of 2^n disjoint closed interval each of length (or Lebesgue measire) equals 3^{-n}. Hence, $\lambda(E_n) = 2^n 3^{-n} = (\frac{2}{3})^n$. Since λ on $[0,1]$ is a finite measure, and E_n is a sequence decreasing to C, we have by Theorem 4.4 that $\lambda(C) = \lim_{n\to\infty} \lambda(E_n) = \lim_{n\to\infty} (\frac{2}{3})^n = 0$.

(v) Show that C does not contain any non-empty open interval, and that the interior of C is empty.

Proof: The proof is by contradiction. Suppose that C contains a non-empty open interval (a,b). Let $\epsilon = b - a$. Notice that $\epsilon > 0$ since (a,b) is non-empty. Choose N sufficiently large so that $3^{-n} < \epsilon$ for all $n \ge N$. Since E_n consists of 2^{-n} disjoint intervals each of length 3^{-n}, then (a,b) cannot be contained in any of these intervals if $n \ge N$ ((a,b) has length bigger than 3^{-n}), and hence (a,b) cannot be contained in E_n for all $n \ge N$. Therefore (a,b) cannot be contained in C, a contradiction. Finally, if $x \in C$ is an interior point, then C must contain an open interval containing x, but from the above, this is impossible. Hence C contains no interior points, in other words, the interior of C is empty.

(vi) Show that every point $x \in [0,1]$ has a ternary expansion of the form $x = \sum_{i=1}^{\infty} \frac{a_i}{3^i}$ with $a_i \in \{0,1,2\}$, which we also denote by $x = 0.a_1, a_2, \ldots$. Conclude that C consists of points whose ternary expansion does not contain the digit 1, i.e., $a_i \in \{0,2\}$.

Proof: We first show how the ternary expansion is obtained. To determine the first digit, we divide the unit interval into three intervals of equal length (namely, $[0,1/3]$, $(1/3,2/3)$, $[2/3,1]$). Points in the left interval are assigned $a_1 = 0$, points in the middle piece are assigned $a_1 = 1$ and points in the right piece are assigned $a_1 = 2$. To determine the second digit a_2, we divide each of the intervals in the previous stage into three intervals of equal length (namely $1/9$). The left piece is assigned $a_2 = 0$, the middle piece is assigned $a_2 = 1$ and the right piece is assigned $a_2 = 2$. This process is repeated indefinitely, and at stage n the digit a_n is determined according to its location in the left piece ($a_n = 0$), or the middle piece ($a_n = 1$), or the right piece ($a_n = 2$). In this way, we see that each point in $[0,1]$

has a ternary expansion. In fact all points have a unique ternary expansion except for points of the form $k/3^n$, these have exactly two ternary expansion. For example $1/3 = 0.100000\ldots = 0.022222\ldots$. In the above description, the middle piece is chosen to be open, hence in the above procedure the ternary expansion obtained for $1/3$ is $0.022222\ldots$.

By the construction of the Cantor set, we see that at stage 1, the middle piece is removed, i.e. all points whose ternary expansion starts with 1 are removed, and only points with $a_1 \in \{0, 2\}$ remain. At stage 2, we remove from the remaining pieces the middle piece of each interval, i.e., we remove all points whose second ternary digit $a_2 = 2$. So only points whose ternary expansion starting with $a_1, a_2 \in \{0, 2\}$ remain. If we continue this process, we see that at stage n, only points whose first n digits $a_1, a_2, \ldots a_n \in \{0, 2\}$ remain. The limiting process, namely C, consists then of all points whose ternary expansion does not contain the digit 1.

(vii) Show that C is uncountable (yet $\lambda(C) = 0$).

Proof: For this we need to define a bijection from C onto $[0, 1]$. Define $f : C \to [0, 1]$ as follows: let $x \in C$, then x has a ternary expansion of the form $x = \sum_{i=1}^{\infty} \frac{a_i}{3^i}$ with $a_i \in \{0, 2\}$. Set $f(x) = f(\sum_{i=1}^{\infty} \frac{a_i}{3^i}) = \sum_{i=1}^{\infty} \frac{b_i}{2^i}$ with $b_i = a_i/2$. Then, it is easy to check that f is a bijection. Since $[0, 1]$ is uncountable, it follows that C is uncountable.