Solutions Extra Practice Final Measure and Integration 2014-15

(1) Consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$ and let $f \in \mathcal{L}^1(\lambda)$ is bounded. Show that the function g defined by

$$g(t) = \int_{\mathbb{R}} \frac{f(x)}{x^2+t^2}\, d\lambda(x),$$

is bounded and continuous on the interval $(1, \infty)$

Proof: Let $f_t(x) = \dfrac{f(x)}{x^2+t^2}$, then for any $x \in \mathbb{R}$ and any $t > 1$ one has $|f_t(x)| \leq |f(x)|$. Hence, for $t > 1$

$$|g(t)| \leq \int_{\mathbb{R}} |f_t(x)|\, d\lambda(x) \leq \int_{\mathbb{R}} |f(x)|\, d\lambda(x) < \infty,$$

Hence g is bounded on $(1, \infty)$. Now $t \in (1, \infty)$ and $(t_n)_n \subseteq (1, \infty)$ such that $\lim_{n\to\infty} t_n = t$. Note that for each $x \in \mathbb{R}$, the function $t \to \dfrac{f(x)}{x^2+t^2}$ is continuous on $(1, \infty)$, hence $\lim_{n\to\infty} f_{t_n}(x) = f_t(x)$. Since $|f_t| \leq |f|$ and $f \in \mathcal{L}^(\lambda)$, then by Lebesgue Dominated Convergence Theorem we have

$$\lim_{n\to\infty} g(t_n)\, d\lambda = \int_{\mathbb{R}} \lim_{n\to\infty} f_{t_n}\, d\lambda = \int_{\mathbb{R}} f_t\, d\lambda = g(t).$$

(2) Consider the measure space $([0,1]\mathcal{B}([0,1]), \lambda)$, where $\mathcal{B}([0,1])$ is the restriction of the Borel σ-algebra to $[0,1]$, and λ is the restriction of Lebesgue measure to $[0,1]$. Let $E_1, \cdots, E_m$ be a collection of Borel measurable subsets of $[0,1]$ such that every element $x \in [0,1]$ belongs to at least n sets in the collection $\{E_j\}_{j=1}^m$, where $n \leq m$. Show that there exists a $j \in \{1, \cdots, m\}$ such that $\lambda(E_j) \geq \dfrac{n}{m}$.

Solution: By hypothesis, for any $x \in [0,1]$ we have $\sum_{j=}^{m} \mathbf{1}_{E_j}(x) \geq n$. Assume for the sake of getting a contradiction that $\lambda(E_j) < \dfrac{n}{m}$ for all $1 \leq j \leq m$. Then,

$$n = \int_{[0,1]} n\, d\lambda \leq \int \sum_{j=}^{m} \mathbf{1}_{E_j}(x)\, d\lambda = \sum_{j=1}^{m} \lambda(E_j) < \sum_{j=}^{m} \frac{n}{m} = n,$$

a contradiction. Hence, there exists $j \in \{1, \cdots, m\}$ such that $\lambda(E_j) \geq \dfrac{n}{m}$.

(3) Let $(X, \mathcal{F}, \mu)$ be a measure space, and $1 < p, q < \infty$ conjugate numbers, i.e. $1/p + 1/q = 1$. Show that if $f \in \mathcal{L}^p(\mu)$, then there exists $g \in \mathcal{L}^q(\mu)$ such that $||g||_q = 1$ and $\int fg\, d\mu = ||f||_p$.

Solution: Note that $q(p-1) = p$, so we define $g = \operatorname{sgn}(f)\left(\dfrac{f}{||f||_p}\right)^{p-1}$. Then,

$$\int |g|^q\, d\mu = \int \frac{|f|^p}{||f||_p^p}\, d\mu = 1.$$

So $||g||_q = 1$ and

$$\int fg\, d\mu = \int |fg|\, d\mu = \int \frac{|f|^p}{||f||_p^{p-1}}\, d\mu = ||f||_p.$$

(4) Consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$, where $\mathcal{B}(\mathbb{R}))$ is the Borel σ-algebra and λ is Lebesgue measure. Let $f \in \mathcal{L}^1(\lambda)$ and define for $h > 0$, the function $f_h(x) = \dfrac{1}{h} \displaystyle\int_{[x,x+h]} f(t)\, d\lambda(t)$.

(a) Show that f_h is Borel measurable for all $h > 0$.
(b) Show that $f_h \in \mathcal{L}^1(\lambda)$ and $||f_h||_1 \leq ||f||_1$.

Solution (a): For $h > 0$, define $u_h(t,x) = \frac{1}{h}\mathbf{1}_{[x,x+h]}(t)f(t)$, then u_h is $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$ measurable. Applying Tonelli's Theorem (Theorem 13.8(ii)) to the positive and negative parts of the function u_h, we have that the functions

$$x \to \int u^+(t,x)\, d\lambda(t) = f_h^+(x), \text{ and } x \to \int u^-(t,x)\, d\lambda(t) = f_h^-(x)$$

are $\mathcal{B}(\mathbb{R})$ measurable. Hence, f_h is Borel measurable for all $h > 0$.

Solution (b): Note that

$$\int \int \frac{1}{h}\mathbf{1}_{[x,x+h]}(t)|f(t)|\, d\lambda(x)\, d\lambda(t) = \int \int \frac{1}{h}\mathbf{1}_{[t-h,t]}(x)|f(t)|\, d\lambda(x)\, d\lambda(t) = \int |f(t)|\, d\lambda(t) < \infty.$$

Hence, by Fubini's Theorem $f_h \in \mathcal{L}^1(\lambda)$ and

$$\int |f_h(x)|\, d\lambda(x) = \int \int \frac{1}{h}\mathbf{1}_{[x,x+h]}(t)|f(t)|\, d\lambda(x)\, d\lambda(t) = \int |f(x)|\, d\lambda(x) = ||f||_1.$$

(5) Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space and (A_i) a sequence in $\mathcal{A}$ such that $\lim_{n\to\infty} \mu(A_n) = 0$.
  (a) Show that $\mathbf{1}_{A_n} \xrightarrow{\mu} 0$, i.e. the sequence $(\mathbf{1}_{A_n})$ converges to 0 in measure.
  (b) Show that for any $u \in \mathcal{L}^1(\mu)$, one has $u\mathbf{1}_{A_n} \xrightarrow{\mu} 0$.
  (c) Show that for any $u \in \mathcal{L}^1(\mu)$, one has

$$\sup_n \int_{\{|u|\mathbf{1}_{A_n} > |u|\}} |u|\mathbf{1}_{A_n}\, d\mu = 0.$$

  (d) Show that $\lim_{n\to\infty} \int_{A_n} u\, d\mu = 0$.

Proof (a): For any $0 < \epsilon < 1$ and any$A \in \mathcal{A}$ with $\mu(A) < \infty$, we have

$$\mu(A \cap \{\mathbf{1}_{A_n} > \epsilon\}) = \mu(A \cap A_n) \le \mu(A_n).$$

Thus, $\limsup_{n\to\infty} \mu(A \cap \{\mathbf{1}_{A_n} > \epsilon\}) = 0$ and hence $\lim_{n\to\infty} \mu(A \cap \{\mathbf{1}_{A_n} > \epsilon\}) = 0$. This implies $\mathbf{1}_{A_n} \xrightarrow{\mu} 0$.

Proof (b): Let $u \in \mathcal{L}^1(\mu)$. For any $\epsilon > 0$ and any$A \in \mathcal{A}$ with $\mu(A) < \infty$, one has

$$\mu(A \cap \{|u|\mathbf{1}_{A_n} > \epsilon\}) = \mu(A \cap A_n \cap \{|u| > \epsilon\}) \le \mu(A_n).$$

This shows that $\lim_{n\to\infty} \mu(A \cap \{|u|\mathbf{1}_{A_n} > \epsilon\}) = 0$, and hence $u\mathbf{1}_{A_n} \xrightarrow{\mu} 0$.

Proof (c): Let $u \in \mathcal{L}^1(\mu)$. Note that $|u|\mathbf{1}_{A_n} \le |u|$, thus the set $\{|u|\mathbf{1}_{A_n} > |u|\}$ is empty. By Theorem 10.9(ii), we have

$$\int_{\{|u|\mathbf{1}_{A_n} > |u|\}} |u|\mathbf{1}_{A_n}\, d\mu = 0$$

for all n and hence $\sup_j \int_{\{|u|\mathbf{1}_{A_n} > |u|\}} |u|\mathbf{1}_{A_n}\, d\mu = 0$.

(6) Let $(X, \mathcal{A}, \mu)$ be a measure space. Show that μ is σ-finite if and only if there exists $f \in L^1(\mu)$ which is **strictly** positive.

Proof: Assume there exists a function $f \in L^1(\mu)$ which is **strictly** positive. We need to find an exhausting sequence $F_n \uparrow X$ such that $\mu(F_n) < \infty$. To this end, let $F_n = \{x \in X : f(x) \ge \frac{1}{n}\}$. Then, $\{F_n\}$ is an increasing sequence of measurable sets such that $X = \{x \in X : f(x) > 0\} = \bigcup_{n=1}^{\infty} F_n$. By the Markov inequality $\mu(F_n) \le n \int f\, d\mu < \infty$. Thus, μ is σ-finite.

Conversely, suppose μ is σ-finite and let (F_n) be an exhausting sequence of finite measure. Define $E_1 = F_1$, and $E_n = F_n \setminus F_{n-1}$ for $n \ge 2$. Then, $(E_n)_n$ is a disjoint sequence of measurable sets such that $\bigcup_{n=1}^{\infty} E_n = X$ and $\mu(E_n) < \infty$ for all $n \ge 1$. Define a function f on X by

$$f(x) = \sum_{n=1}^{\infty} \frac{2^{-n}}{\mu(E_n) + 1}\mathbf{1}_{E_n}(x).$$

Then, $f > 0$, and by Beppo-Levi,

$$\int_X f\,d\mu = \sum_{n=1}^{\infty} \frac{2^{-n}}{\mu(E_n)+1} \int_X \mathbf{1}_{E_n}\,d\mu = \sum_{n=1}^{\infty} \frac{2^{-n}\mu(E_n)}{\mu(E_n)+1} \leq \sum_{n=1}^{\infty} 2^{-n} = 1 < \infty.$$

So that $f \in \mathcal{L}^1(\mu)$ is the required function.