Practice Final Measure and Integration I 12015-16

1. Let μ and ν be two measures on the measure space $(E, \mathcal{B})$ such that $\mu(A) \leq \nu(A)$ for all $A \in \mathcal{B}$.

   (a) Show that if f is any non-negative measurable function on $(E, \mathcal{B})$, then $\int_E f\, d\mu \leq \int_E f\, d\nu$.

   (b) Prove that if ν is a finite measure, then $\mathcal{L}^2(\nu) \subseteq \mathcal{L}^1(\mu)$.

   Proof (a) Suppose first that $f = 1_A$ is the indicator function of some set $A \in \mathcal{B}$. Then

   $$\int_E f\, d\mu = \mu(A) \leq \nu(A) = \int_E f\, d\nu.$$

   Suppose now that $f = \sum_{k=1}^{n} \alpha_k 1_{A_k}$ is a non-negative measurable step function. Then,

   $$\int_E f\, d\mu = \sum_{k=1}^{n} \alpha_k \mu(A_k) \leq \sum_{k=1}^{n} \alpha_k \nu(A_k) = \int_E f\, d\nu.$$

   Finally, let f be a non-negative measurable function, then there exists a sequence of non-negative measurable step functions f_n such that $f_n \uparrow f$. By Beppo-Levi,

   $$\int_E f\, d\mu = \lim_{n\to\infty} \int_E f_n\, d\mu \leq \lim_{n\to\infty} \int_E f_n\, d\nu = \int_E f\, d\nu.$$

   Proof (b) From part (a) we see that if $f \in \mathcal{L}^1(\nu)$, then $f \in \mathcal{L}^1(\mu)$, i.e. $\mathcal{L}^1(\nu) \subseteq \mathcal{L}^1(\mu)$. If ν is a finite measure, then by Exercise 12.1 (ii) and the above, we have $\mathcal{L}^2(\nu) \subseteq \mathcal{L}^1(\nu) \subseteq L^1(\mu)$.

2. Consider the measure space $((0,1], \mathcal{B}((0,1]), \lambda)$, where $\mathcal{B}((0,1])$ and λ are the restrictions of the Borel σ-algebra and Lebesgue measure to the interval $(0,1]$. Determine the value of

   $$\lim_{n\to\infty} \int_{(0,1]} e^{1/x}(1+n^2x)^{-1} \sin(ne^{-1/x}\, d\lambda(x).$$

   Proof: Let $u_n(x) = e^{1/x}(1+n^2x)^{-1}\sin(ne^{-1/x}$, then $\lim_{n\to\infty} u_n(x) = 0$ for all $x \in (0,1]$. Since $|\sin y| \leq y$ for all $y \geq 0$, we have

   $$|u_n(x)| \leq e^{1/x}(1+n^2x)^{-1} ne^{-1/x} = \frac{n}{1+n^2x} = \frac{1}{\sqrt{x}} \cdot \frac{n\sqrt{x}}{1+n^2x} \leq \frac{1}{\sqrt{x}}.$$

Since the function $\dfrac{1}{\sqrt{x}}$ is positive, measurable and the improper Riemann integrable on $(0,1]$ exists, it follows that it is Lebesgue integrable on $(0,1]$. By Lebesgue Dominated Convergence Theorem, we have

$$\begin{aligned}\lim_{n\to\infty}\int_{(0,1]} e^{1/x}(1+n^2x)^{-1}\sin(ne^{-1/x}\,d\lambda(x) &= \lim_{n\to\infty}\int u_n(x)\,d\lambda(x)\\ &= \int \lim_{n\to\infty} u_n(x)\,d\lambda(x) = 0.\end{aligned}$$

3. Let $(X,\mathcal{F},\mu)$ be a **finite** measure space. Assume $f\in\mathcal{L}^2(\mu)$ satisfies $0<||f||_2<\infty$, and let $A=\{x\in X : f(x)\neq 0\}$. Show that

$$\mu(A)\geq \frac{(\int f\,d\mu)^2}{\int f^2\,d\mu}.$$

Solution: Since $f=0$ on A^c, we have $\int f\,d\mu=\int f\mathbf{1}_A\,d\mu$. Since μ is a finite measure and $(\mathbf{1}_A)^2=\mathbf{1}_A$, then

$$||\mathbf{1}_A||_2=(\mu(A))^{1/2}<\infty.$$

Thus, $\mathbf{1}_A\in\mathcal{L}^2(\mu)$ and by Hölder's inequality

$$\int f\,d\mu\leq ||f||_2||\mathbf{1}_A||_2=|f||_2(\mu(A))^{1/2}.$$

Squaring both sides and dividing by

$$||f||_2^2=\int f^2\,d\mu\,(>0),$$

we get

$$\mu(A)\geq \frac{(\int f\,d\mu)^2}{\int f^2\,d\mu}.$$

4. Let $E=\{(x,y) : y<x<1, ,0<y<1\}$. We consider on E the restriction of the product Borel σ-algebra, and the restriction of the product Lebesgue measure $\lambda\times\lambda$. Let $f:E\to\mathbb{R}$ be given by $f(x,y)=x^{-3/2}\cos(\frac{\pi y}{2x})$.

   (a) Show that f is $\lambda\times\lambda$ integrable on E.

   (b) Define $F:(0,1)\to\mathbb{R}$ by $F(y)=\int_{(y,1)} x^{-3/2}\cos(\frac{\pi y}{2x})\,d\lambda(x)$. Determine the value of

$$\int F(y)\,d\lambda(y).$$

Solution (a) : Notice that f is continuous, and hence measurable. Furthermore, $|f(x,y)| \le x^{-3/2}$. The function $g(x,y) = x^{-3/2}$ is non-negative and measurable on E, hence by Tonelli's Theorem,

$$\begin{aligned} \int_E |f(x,y)|\, d(\lambda \times \lambda)(x,y) &\le \int_E g(x,y)\, d(\lambda \times \lambda)(x,y) \\ &= \int_0^1 \int_0^x x^{-3/2}\, dy\, dx \\ &= \int_0^1 x^{-1/2}\, dx = 2. \end{aligned}$$

Notice that the integrands are Riemann integrable, hence the Riemann integral equals the Lebesgue integral. This shows that f is $\lambda \times \lambda$ integrable on E.

Solution (b) : By Fubini's Theorem

$$\int \int f(x,y)\, d\lambda(x)\, d\lambda(y) = \int \int f(x,y)\, d\lambda(y)\, d\lambda(x).$$

Notice that for each fixed $0 < x < 1$, the function $f(x,y)$ is Riemann-integrable in y on the interval $(0,x)$ and

$$\int_0^x x^{-3/2} \cos(\frac{\pi y}{2x})\, dy = \frac{2}{\pi} x^{-1/2},$$

and the function $\frac{2}{\pi} x^{-1/2}$ is Riemann-integrable in x on the interval $(0,1)$, and

$$\int_0^1 \frac{2}{\pi} x^{-1/2}\, dx = \frac{4}{\pi}.$$

Thus,

$$\int F(y)\, d\lambda(y) = \int \int f(x,y)\, d\lambda(x)\, d\lambda(y) = \int_0^1 \int_0^x x^{-3/2} \cos(\frac{\pi y}{2x})\, dy\, dx = \frac{4}{\pi}.$$

5. Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space, and (f_j) a uniformly integrable sequence of measurable functions. Define $F_k = \sup_{1 \le j \le k} |f_j|$ for $k \ge 1$.

   (a) Show that for any $w \in \mathcal{M}^+(\mathcal{A})$,

   $$\int_{\{F_k > w\}} F_k\, d\mu \le \sum_{j=1}^{k} \int_{\{|f_j| > w\}} |f_j|\, d\mu.$$

   (b) Show that for every $\epsilon > 0$, there exists a $w_\epsilon \in \mathcal{L}^1_+(\mu)$ such that for all $k \ge 1$

   $$\int_X F_k\, d\mu \le \int_X w_\epsilon\, d\mu + k\epsilon.$$

(c) Show that

$$\lim_{k\to\infty} \frac{1}{k} \int_X F_k \, d\mu = 0.$$

Proof (a) Let $w \in \mathcal{M}^+(\mathcal{A})$, then

$$\begin{aligned}\int_{\{F_k > w\}} F_k \, d\mu &\leq \sum_{j=1}^k \int_{\{F_k > w\} \cap \{|f_j| = F_k\}} F_k \, d\mu \\ &\leq \sum_{j=1}^k \int_{\{|f_j| > w\}} |f_j.| \, d\mu.\end{aligned}$$

Proof (b) Let $\epsilon > 0$. By uniform integrability of the sequence (f_j) there exists $w_\epsilon \in \mathcal{L}^+(\mu)$ such that

$$\int_{\{|f_j| > w_\epsilon\}} |f_j| \, d\mu < \epsilon$$

for all $j \geq 1$. By part (a)

$$\int_{\{F_k > w_\epsilon\}} F_k \, d\mu \leq \sum_{j=1}^k \int_{\{|f_j| > w_\epsilon\}} |f_j| \, d\mu \leq k\epsilon.$$

Now,

$$\begin{aligned}\int_X F_k \, d\mu &= \int_{\{F_k > w_\epsilon\}} F_k \, d\mu + \int_{\{F_k \leq w_\epsilon\}} F_k \, d\mu \\ &\leq k\epsilon + \int_X w_\epsilon \, d\mu.\end{aligned}$$

Proof (c) For any $\epsilon > 0$, by part (b),

$$\frac{1}{k} \int_X F_k \, d\mu \leq \frac{1}{k} \int_X w_\epsilon \, d\mu + \epsilon.$$

Thus,

$$\limsup_{k\to\infty} \frac{1}{k} \int_X F_k \, d\mu \leq \epsilon,$$

for any ϵ. Since $F_k \geq 0$, we see that

$$\limsup_{k\to\infty} \frac{1}{k} \int_X F_k \, d\mu = \lim_{k\to\infty} \frac{1}{k} \int_X F_k \, d\mu = 0.$$

6. Suppose μ and ν are finite measures on the measurable space $(X, \mathcal{A})$ which have the same null sets. Show that there exists a measurable function f such that $0 < f < \infty$ μ a.e. and ν a.e. and for all $A \in \mathcal{A}$ one has

$$\nu(A) = \int_A f \, d\mu \quad \text{and} \quad \mu(A) = \int_A \frac{1}{f} \, d\nu.$$

Proof The fact that μ and ν have the same null sets implies that $\nu \ll \mu$ and $\mu \ll \nu$ (in fact in this case we refer to μ and ν as equivalent measure). So the notions μ a.e. and ν a.e. are the same. By Radon-Nikodym Theorem there exist $f \in \mathcal{L}^1_+(\mu)$ and $g \in \mathcal{L}^1_+(\nu)$ such that for all $A \in \mathcal{A}$,

$$\nu(A) = \int_A f\, d\mu \quad \text{and } \mu(A) = \int_A g\, d\nu.$$

Furthermore, the functions f and g are unique μ and ν a.e. By Exercise 1 of the last set of exercises exercisesRadonNikodym.pdf, we have for any $A \in \mathcal{A}$,

$$\nu(A) = \int_A 1\, d\nu = \int_A f\, d\mu = \int_A fg\, d\nu.$$

By Corollary 10.14(i) this implies that $1 = fg$ ν and hence μ a.e. From this and Corollary 10.13 we conclude that $0 < f < \infty$ and $g = 1/f$ μ and ν a.e.