Measure and Integration: Exercise on Radon-Nikodym Theorem, 2014-15

1. Let $(E, \mathcal{B}, \nu)$ be a measure space, and $h : E \to \mathbb{R}$ a non-negative measurable function. Define a measure μ on $(E, \mathcal{B})$ by $\mu(A) = \int_A h d\nu$ for $A \in \mathcal{B}$. Show that for every non-negative measurable function $F : E \to \mathbb{R}$ one has

$$\int_E F\, d\mu = \int_E Fh\, d\nu.$$

Conclude that the result is still true for $F \in \mathcal{L}^1(\mu)$ which is not necessarily non-negative.

Proof Suppose first that $F = 1_A$ is the indicator function of some measurable set $A \in \mathcal{B}$. Then,

$$\int_E F\, d\mu = \mu(A) = \int_A h\, d\nu = \int_E 1_A h d\nu = \int_E F h d\nu.$$

Suppose now that $F = \sum_{k=1}^{n} \alpha_k 1_{A_k}$ is a non-negative measurable step function. Then,

$$\int_E F\, d\mu = \sum_{k=1}^{n} \alpha_k \mu(A_k) = \sum_{k=1}^{n} \alpha_k \int_E 1_A h d\nu = \int_E \sum_{k=1}^{n} \alpha_k 1_A h d\nu = \int_E F h d\nu.$$

Suppose that F is a non-negative measurable function, then there exists a sequence of non-negative measurable step functions F_n such that $F_n \uparrow F$. Then, $F_n h \uparrow F h$, and by Beppo-Levi,

$$\int_E F\, d\mu = \lim_{n\to\infty} \int_E F_n\, d\mu = \lim_{n\to\infty} \int_E F_n h d\nu = \int_E F h d\nu.$$

Finally, suppose that $F \in \mathcal{L}^1(\mu)$. Since F^+, F^- are non-negative, we have

$$\int_E F^+\, d\mu = \int_E F^+ h\, d\nu \text{ and } \int_E F^-\, d\mu = \int_E F^- h\, d\nu.$$

Since $F \in \mathcal{L}^1(\mu)$, from the above we see that $Fh \in \mathcal{L}^1(\nu)$, hence

$$\int_E F\, d\mu = \int_E F^+\, d\mu - \int_E F^-\, d\mu = \int_E F^+ h\, d\nu - \int_E F^- h\, d\nu = \int_E F h\, d\nu.$$

2. Let $(X, \mathcal{B}, \nu)$ be a measure space, and suppose $X = \bigcup_{n=1}^{\infty} E_n$, where $\{E_n\}$ is a collection of pairwise disjoint measurable sets such that $\nu(E_n) < \infty$ for all $n \geq 1$. Define μ on $\mathcal{B}$ by $\mu(B) = \sum_{n=1}^{\infty} 2^{-n} \nu(B \cap E_n)/(\nu(E_n) + 1)$.

(a) Prove that μ is a finite measure on $(X, \mathcal{B})$.

(b) Let $B \in \mathcal{B}$. Prove that $\mu(B) = 0$ **if and only if** $\nu(B) = 0$.

(c) Find explicitly two positive integrable functions f and g such that

$$\mu(A) = \int_A f\, d\nu \text{ and } \nu(A) = \int_A g\, d\mu,$$

for all $A \in \mathcal{B}$.

Proof (a): Clearly $\mu(\emptyset) = 0$, and

$$\mu(X) = \sum_{n=1}^{\infty} 2^{-n}\nu(E_n)/(\nu(E_n)+1) \le \sum_{n=1}^{\infty} 2^{-n} = 1 < \infty.$$

Now, let (C_n) be a disjoint sequence in $\mathcal{B}$. Then,

$$\begin{aligned}
\mu(\textstyle\bigcup_{m=1}^{\infty} C_m) &= \sum_{n=1}^{\infty} 2^{-n}\nu((\bigcup_{m=1}^{\infty} C_m) \cap E_n)/(\nu(E_n)+1) \\
&= \sum_{n=1}^{\infty} 2^{-n} \sum_{m=1}^{\infty} \nu(C_m \cap E_n)/(\nu(E_n)+1) \\
&= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} 2^{-n}\nu(C_m \cap E_n)/(\nu(E_n)+1) \\
&= \sum_{m=1}^{\infty} \mu(C_m).
\end{aligned}$$

Thus, μ is a finite measure.

Proof (b): Suppose that $\nu(B) = 0$, then $\nu(B \cap E_n) = 0$ for all n, hence $\mu(B) = 0$. Conversely, suppose $\mu(B) = 0$, then $\nu(B \cap E_n) = 0$ for all n. Since $X = \bigcup_{n=1}^{\infty} E_n$ (disjoint union), then

$$\nu(B) = \nu(B \cap \bigcup_{n=1}^{\infty} E_n) = \nu(\bigcup_{n=1}^{\infty}(B \cap E_n)) = \sum_{n=1}^{\infty} \nu(B \cap E_n) = 0.$$

Proof (c): By (b), we have $\mu \ll \nu$ and $\nu \ll \nu$, so we are looking for the Radon Nikokodym derivatives of μ with respect to ν and of ν with respect to μ. Let $f = \sum_{n=1}^{\infty} \frac{2^{-n}}{\nu(E_n)+1}\mathbf{1}_{E_n}$. Then, $f > 0$ and

$$\int f\, d\nu = \sum_{n=1}^{\infty} 2^{-n}\nu(B \cap E_n)/(\nu(E_n)+1) = \mu(A) is \le \sum_{n=1}^{\infty} 2^{-n} = 1,$$

hence, $f \in \mathcal{L}^1(\nu)$ is one of the required Radon Nikodym derivatives.. Let $g = 1/f$. Since $f > 0$ and measurable then so is $1/f$. Furthermore, for any $A \in \mathcal{B}$, and by exercise 1,

$$\nu(A) = \int_A f\frac{1}{f}\, d\nu = \int \frac{1}{f}\, d\mu.$$

Thus, $g = 1/f$ is the second required Radon Nikodym derivative.

3. Suppose μ, ν and λ are finite measures on $(X, \mathcal{B})$ such that $\mu \ll \nu$ and $\nu \ll \lambda$. Show that $\mu \ll \lambda$ and $\dfrac{d\mu}{d\lambda} = \dfrac{d\mu}{d\nu} \cdot \dfrac{d\nu}{d\lambda}$ λ a.e.

   Proof: Suppose $A \in \mathcal{B}$ satisfies $\lambda(A) = 0$. Since $\nu \ll \lambda$, then $nu(A) = 0$ and since $\mu \ll \nu$ we get $\mu(A) = 0$. Thus, $\mu \ll \lambda$. Again using exercise 1, we have for any $B \in \mathcal{B}$,
$$\int_B \frac{d\mu}{d\nu} \cdot \frac{d\nu}{d\lambda}\, d\lambda = \int_B \frac{d\mu}{d\nu}\, d\nu = \mu(B) = \int_B \frac{d\mu}{d\lambda}\, d\lambda.$$
   By the uniqueness of the Radon-Nikodym derivative, we have $\dfrac{d\mu}{d\lambda} = \dfrac{d\mu}{d\nu} \cdot \dfrac{d\nu}{d\lambda}$ λ a.e.

4. Suppose that μ_i, ν_i are finite measures on $(X, \mathcal{A})$ with $\mu_i \ll \nu_i$ for $i = 1, 2$. Let $\nu = \nu_1 \times \nu_2$ and $\mu = \mu_1 \times \mu_2$ be the corresponding product measures on $(X \times X, \mathcal{A} \otimes \mathcal{A})$.

   (a) Show that $\mu \ll \nu$.

   (b) Prove that $\dfrac{d\mu}{d\nu}(x, y) = \dfrac{d\mu_1}{d\nu_1}(x) \cdot \dfrac{d\mu_2}{d\nu_2}(y)$ ν a.e.

   Proof(a): For $E \in \mathcal{A} \otimes \mathcal{A}$ and $x \in X$, let $E_x = \{y \in X : (x, y) \in E\}$. Then, by Theorem 13.5 the functions $x \to \mu_2(E_x)$ and $x \to \nu_2(E_x)$ are $\mathcal{A}$ measurable, and
$$\nu(E) = \int_X \nu_2(E_x)\, d\nu_1(x), \text{ and } \mu(E) = \int_X \mu_2(E_x)\, d\mu_1(x).$$
   Assume $\nu(E) = 0$, then by Theorem 10.9(i) we have $\nu_2(E_x) = 0$ ν_1 a.e. Since $\mu_2 \ll \nu_2$, we have $\mu_2(E_x) = 0$ ν_1 a.e Since $\mu_1 \ll \nu_1$, we get $\mu_2(E_x) = 0$ μ_1 a.e Again by Theorem 10.9(i), we have
$$\mu(E) = \int_X \mu_2(E_x)\, d\mu_1(x) = 0.$$
   Therefore, $\mu \ll \nu$.

   Proof(b): Let $E \in \mathcal{A} \otimes \mathcal{A}$, then by the Radon Nikodym Theorem $\frac{d\mu}{d\nu}$ unique ν a.e. function such that $\mu(E) = \int_E \frac{d\mu}{d\nu}(x, y)\, d\nu(x, y)$. By Exercise 1, and Theorem 13.5
$$\begin{aligned}
\mu(E) &= \int\int \mathbf{1}_{E_x}(y)\, d\mu_2(y)\, d\mu_1(x) \\
&= \int (\int \mathbf{1}_{E_x}(y) \frac{d\mu_2}{d\nu_2}(y) d\nu_2(y))\, d\mu_1(x) \\
&= \int (\int \mathbf{1}_{E_x}(y) \frac{d\mu_2}{d\nu_2}(y) d\nu_2(y)) \frac{d\mu_1}{d\nu_1}(x)\, d\nu_1(x) \\
&= \int\int \mathbf{1}_E(x, y) \frac{d\mu_2}{d\nu_2}(y) \frac{d\mu_1}{d\nu_1}(x)\, d\nu_2(y) d\nu_1(x) \\
&= \int_E \frac{d\mu_2}{d\nu_2}(y) \frac{d\mu_1}{d\nu_1}(x)\, d\nu(x, y).
\end{aligned}$$
   By the uniqueness of the Radon Nikodym Derivative we have
$$\frac{d\mu}{d\nu}(x, y) = \frac{d\mu_1}{d\nu_1}(x) \cdot \frac{d\mu_2}{d\nu_2}(y)$$
   ν a.e.