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1. Consider the binomial model with $u = 2^1$, $d = 2^{-1}$, and $r = 1/4$, and consider a perpetual American put option with $S_0 = 10$ and $K = 12$. Suppose that Alice and Bob each buy such an option

   (a) Suppose that Alice uses the strategy of exercising the first time the price reaches 5 euros. What should then the price be at time 0?

   (b) Suppose that Bob uses the strategy of exercising the first time the price reaches 2.5 euros. What should then the price be at time 0?

   (c) What is the probability that the price reaches 20 euros for the first time at time $n = 5$?

Solution (a): Note that the price process has the form $S_n = S_0 2^{M_n} = 10 2^{M_n}$, where M_n is the symmetric random walk. The strategy of Alice corresponds to the exercise policy τ_{-1}, the first time the random walk reaches level -1. Using formula (5.2.13), we find

$$\widetilde{E}\left[\left(\frac{1}{1+r}\right)^{\tau_{-1}}\right] = \widetilde{E}\left[\left(\frac{4}{5}\right)^{\tau_{-1}}\right] = \frac{1}{2}.$$

Thus, the price at time 0 is

$$V(\tau_{-1}) = \widetilde{E}\left[\left(\frac{1}{1+r}\right)^{\tau_{-1}}(K - S_{\tau_{-1}})\right] = (12-5)/2 = 7/2 = 3.5.$$

Solution (b): The strategy of Bob corresponds to the exercise policy τ_{-2}, the first time the random walk reaches level -2. Note that

$$\widetilde{E}\left[\left(\frac{1}{1+r}\right)^{\tau_{-2}}\right] = \widetilde{E}\left[\left(\frac{4}{5}\right)^{\tau_{-2}}\right] = \frac{1}{4}.$$

Thus, the price at time 0 is

$$V(\tau_{-2}) = \widetilde{E}\left[\left(\frac{1}{1+r}\right)^{\tau_{-2}}(K - S_{\tau_{-2}})\right] = (12-2.5)/4 = 2.375.$$

Solution (c): Using formula (5.2.22) (with $j = 3$), the required probability is

$$P(\tau_1 = 5) = 1/16.$$

2. Let $M_0, M_1, \cdots$ be the symmetric random walk. Define for $a \in Z$, $M_n^a = a + M_n$. The process $M_0^a, M_1^a, \cdots$ is called the symmetric random walk starting in a. Let $b \in Z$ be such that $n + b - a$ is even.

(a) Let $N_n(a, b)$ be the number of paths of length n starting in a and ending in b. Show that $N_n(a,b) = \binom{n}{\frac{1}{2}(n+b-a)}$. Conclude that

$$P(M_n^a = b) = \binom{n}{\frac{1}{2}(n+b-a)} \frac{1}{2^n}.$$

(b) Let $N_n^0(a, b)$ be the number of paths of length n starting in a and ending in b that cross the x-axis at least once. Use the reflection principle to prove that if $a, b > 0$, then $N_n^0(a, b) = N_n(-a, b)$.

(c) Let $b > 0$, using part (b) show that the number of paths of length n starting in 0 which does not cross the x-axis (except at the starting point) equals $\frac{b}{n} N_n(0, b)$.

(d) Use part (c) to prove that if $b > 0$, then

$$P(M_n = b, \min_{1 \le k \le n-1} M_k > 0) = \frac{b}{n} P(M_n = b).$$

Solution (a): Let u be the number of steps to the right, and d the number of steps to the left. Then $u + d = n$, and $u - d = b - a$. Solving, we get $u = \frac{1}{2}(n + b - a)$, and $d = \frac{1}{2}(n - b + a)$. Thus, $N_n(a, b) = \binom{n}{\frac{1}{2}(n+b-a)}$. Finally,

$$P(M_n^a = b) = N_n(a,b)\frac{1}{2^n} = \binom{n}{\frac{1}{2}(n+b-a)} \frac{1}{2^n}.$$

Solution (b): Consider a path of length n starting in $-a$ and ending in b. Such a path must cross the x-axis at some time $k \in \{1, \cdots, n-1\}$. Now, reflect that segment of the path between time 0 and time k to obtain a new path of length n starting in a and ending in b that crosses the x-axis at least once in the time interval $[1, n-1]$. This gives us a one-to-one correspondence between paths of length n starting in $-a$ and ending in b, and paths of length n starting in a and ending in b that cross the x-axis at least once in the time interval $[1, n-1]$. Thus, $N_n^0(a, b) = N_n(-a, b)$.

Solution (c): Since $b > 0$, the first step of any path of length n starting in 0 which does not cross the x-axis (except at time 0) must be to the right. After this step, we must make an additional $n-1$ steps starting in 1 and ending in b, such that during these steps no visit to the x-axis occurs. This can be done in $N_{n-1}(1,b) - N_{n-1}^0(1,b)$ ways. Using part(b), we have

$$N_{n-1}(1,b) - N_{n-1}^0(1,b) = \binom{n-1}{\frac{1}{2}(n-1+b-1)} - \binom{n-1}{\frac{1}{2}(n-1+b+1)}$$

$$
\begin{aligned}
&= \frac{(n-1)!}{(\frac{1}{2}(n+b-2))!(\frac{1}{2}(n-b))!} - \frac{(n-1)!}{(\frac{1}{2}(n-b-2))!(\frac{1}{2}(n+b))!} \\
&= \frac{(n)!}{(\frac{1}{2}(n+b))!(\frac{1}{2}(n-b))!} \cdot \frac{b}{n} \\
&= \frac{b}{n}\binom{n}{\frac{1}{2}(n+b)} \\
&= \frac{b}{n}N_n(0,b).
\end{aligned}
$$

Solution (d): Note that in the event $\{M_n = b, \min_{1\le k\le n-1} M_k > 0\}$ we are looking at paths of length n starting in 0 ending in b, that never crosses the x-axis in the time interval $[1, n-1]$. From part (c), the number of such paths equals $\frac{b}{n}N_n(0,b)$. Since each path has a probability $\frac{1}{2^n}$ of occurring, we have

$$P(M_n = b, \min_{1\le k\le n-1} M_k > 0) = \frac{b}{n}N_n(0,b)\frac{1}{2^n} = \frac{b}{n}P(M_n = b).$$