Universiteit Utrecht — Universiteit Utrecht — Boedapestlaan 6

Mathematisch Instituut — 3584 CD Utrecht

Uitwerkingen OefenDeeltentamen 1 Inleiding Financiele Wiskunde, 2010

1. Consider a 2-period binomial model with $S_0 = 10$, $u = 1.25$, $d = 0.75$, and $r = 0.2$. Suppose the real probability measure P satisfies $P(H) = p = 0.6 = 1 - P(T)$.

   (a) Consider an option with payoff $V_2 = (\max(S_1, S_2) - 11)^+$. Determine the price V_n at time $n = 0, 1$.

   (b) Suppose $\omega_1\omega_2 = HH$, find the values of the portfolio process $\Delta_0, \Delta_1(H)$ so that the corresponding wealth process satisfies $X_2(HH) = V_2(HH)$. Describe the corresponding strategy.

   (c) Determine explicitly the state price density process

   $$\frac{Z_0}{(1+r)^0}, \frac{Z_1}{(1+r)}, \frac{Z_2}{(1+r)^2},$$

   where

   $$Z_2(\omega_1\omega_2) = Z(\omega_1\omega_2) = \frac{\widetilde{P}(\omega_1\omega_2)}{P(\omega_1\omega_2)}$$

   with $\widetilde{P}$ the risk neutral probability measure, and $Z_i = E_i(Z)$, $i = 0, 1$.

   (d) Consider the utility function $U(x) = \ln x^2$. Find a random variable X (which is a function of the two coin tosses) that maximizes $E(U(X))$ subject to the condition that $\widetilde{E}\left(\frac{X}{(1+r)^2}\right) = 10$.

Solution(a) From the given parameters, we have $\widetilde{p} = 0.9 = 1 - \widetilde{q}$. From the price tree and the definition of V_2, we have $V_2(HH) = 4.625$, $V_2(HT) = 1.5$, $V_2(TH) = 0 = V_2(TT)$. Therefore

$$V_1(H) = \frac{1}{1.2}[(0.9)(4.625) + (0.1)(1.5)] = 3.59375,$$

$$V_1(T) = \frac{1}{1.2}[(0.9)(0) + (0.1)(0)] = 0,$$

$$V_0 = \frac{1}{1.2}[(0.9)(3.59375) + (0.1)(0)] = 2.6953125.$$

Solution(b) If $\omega_1\omega_2 = HH$, then

$$\Delta_0 = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)} = \frac{3.59375 - 0}{12.5 - 7.5} = 0.71875,$$

and

$$\Delta_1(H) = \frac{V_2(HH) - V_2(HT)}{S_2(HH) - S_2(HT)} = \frac{4.625 - 1.5}{15.625 - 9.375} = 0.5.$$

Time 0: sell the option for $V_0 = 2.6953125$ Euros and buy 0.71875 of the underlying stock. In order to finance this you need to borrow 4.4921875 from the bank. Therefore.

$$X_0 = 2.6953125 = (0.71875)(10) + (-4.4921875).$$

Time 1: your wealth is now

$$X_1 = (0.71875)(12.5) - (1.2)(4.4921875) = 3.59375.$$

Adjust your portfolio:

$$X_1(H) = 3.59375 = (0.5)(12.5) + (3.59375 - (0.5)(12.5)) = (0.5)(12.5) - 2.65625.$$

Time 2: your wealth is

$$X_2(HH) = (0.5)(15.625) - (1.2)(2.65625) = 4.625 = V_2(HH).$$

Solution(c) $\dfrac{Z_0}{(1+r)^0} = 1$, and

$$\frac{Z_1(H)}{(1+r)^1} = \frac{\widetilde{P}(H)}{P(H)(1.2)} = \frac{5}{4}.$$

$$\frac{Z_1(T)}{(1+r)^1} = \frac{\widetilde{P}(T)}{P(T)(1.2)} = \frac{5}{24}.$$

$$\frac{Z_2(HH)}{(1+r)^2} = \frac{\widetilde{P}(HH)}{P(HH)(1.2)^2} = \frac{25}{16}.$$

$$\frac{Z_2(HT)}{(1+r)^2} = \frac{\widetilde{P}(HT)}{P(HT)(1.2)^2} = \frac{25}{96}.$$

$$\frac{Z_2(TH)}{(1+r)^2} = \frac{\widetilde{P}(TH)}{P(TH)(1.2)^2} = \frac{25}{96}.$$

$$\frac{Z_2(TT)}{(1+r)^2} = \frac{\widetilde{P}(TT)}{P(TT)(1.2)^2} = \frac{25}{576}.$$

Solution(d) Using the same notation as in the book, we let $x_1 = X(HH)$, $x_2 = X(HT)$, $x_3 = X(TH), x_4 = X(TH)$, and note that $P(HH) = \frac{9}{25}$, $P(HT) = \frac{6}{25}$, $P(TH) = \frac{6}{25}$, $P(TT) = \frac{4}{25}$. Our aim is to find a vector (x_1, x_2, x_3, x_4) that maximizes

$$E(U(X)) = \frac{9}{25}\ln x_1^2 + \frac{6}{25}\ln x_2^2 + \frac{6}{25}\ln x_3^2 + \frac{4}{25}\ln x_4^2$$

subject to the condition $\widetilde{E}\left(\frac{X}{(1+r)^2}\right) = 10$, which is equivalent to

$$\frac{9}{25}\frac{25}{16}x_1 + \frac{6}{25}\frac{25}{96}x_2 + \frac{6}{25}\frac{25}{96}x_3 + \frac{4}{25}\frac{25}{576}x_4 = 10.$$

To solve we consider the Lagrangian

$$L = \frac{9}{25}\ln x_1^2 + \frac{6}{25}\ln x_2^2 + \frac{6}{25}\ln x_3^2 + \frac{4}{25}\ln x_4^2 - \lambda\left(\frac{9}{25}\frac{25}{16}x_1 + \frac{6}{25}\frac{25}{96}x_2 + \frac{6}{25}\frac{25}{96}x_3 + \frac{4}{25}\frac{25}{576}x_4 - 10\right).$$

Taking the partial derivatives of L w.r.t. x_1, x_2, x_3, x_4, and setting these to zero (or using directly equations 3.3.23 from the book), we can solve for x_1, x_2, x_3, x_4 in terms of λ. This leads to

$$x_1 = \frac{32}{25\lambda},\ x_2 = \frac{192}{25\lambda},\ x_3 = \frac{192}{25\lambda},\ x_4 = \frac{1152}{25\lambda}.$$

Plugging in these values in the constraint equation, we get

$$\frac{18}{25\lambda} + \frac{12}{25\lambda} + \frac{12}{25\lambda} + \frac{8}{25\lambda} = 10.$$

Solving the above equation, we get $\lambda = \frac{1}{5}$, and hence

$$x_1 = \frac{32}{5},\ x_2 = \frac{192}{5},\ x_3 = \frac{192}{5},\ x_4 = \frac{1152}{5}.$$

2. Consider the N-period binomial model. Consider the random variables $X_1, \ldots, X_N$ on (Ω, P) defined by

$$X_i(\omega_1 \ldots \omega_N) = \begin{cases} 2, & \text{if } \omega_i = H, \\ 0, & \text{if } \omega_i = T. \end{cases}$$

   (a) Assume $P(H) = 1/2 = P(T)$. Let $Z_0 = 1$, and $Z_n = X_1 \ldots X_n$, $n = 1, 2, \ldots, N$. Prove that the process $Z_0, Z_1, \ldots, Z_N$ is a martingale w.r.t. P.

   (b) Suppose $P(H) = 1/4 = 1 - P(T)$. Show that the process $Z_0, Z_1, \ldots, Z_N$ in part (a) is now a supermartingale w.r.t. P, while the process $Z_0^2, Z_1^2, \ldots, Z_N^2$ is a martingale w.r.t. P.

Solution(a) Notice that $Z_{n+1} = Z_n X_{n+1}$ and $E(X_{n+1}) = 1$. Since Z_n is known at time n and X_{n+1} is independent of the first n tosses, we have

$$E_n(Z_{n+1}) = E_n(Z_n X_{n+1}) = Z_n E_n(X_{n+1}) = Z_n E(X_{n+1}) = Z_n.$$

Therefore, $Z_0, Z_1, \ldots, Z_N$ is a martingale w.r.t. P.

Solution(b) If $P(H) = 1/4 = 1 - P(T)$, then $E(X_n) = 1/2$ and $E(X_n^2) = 1$ for all n. Thus,

$$E_n(Z_{n+1}) = E_n(Z_n X_{n+1}) = Z_n E_n(X_{n+1}) = Z_n E(X_{n+1}) = \frac{1}{2} Z_n < Z_n,$$

and

$$E_n(Z_{n+1}^2) = E_n(Z_n^2 X_{n+1}^2) = Z_n^2 E_n(X_{n+1}^2) = Z_n^2 E(X_{n+1}^2) = Z_n^2.$$

Therefore, $Z_0, Z_1, \ldots, Z_N$ is a supermartingale, while $Z_0^2, Z_1^2, \ldots, Z_N^2$ is a martingale w.r.t. P.

3. Consider the N-period binomial model.

   (a) Assume $X_0, X_1, \ldots, X_N$ is a Markov process w.r.t. the risk neutral measure $\widetilde{P}$. Consider an option with payoff $V_N = X_N^2$. Show that for each $n = 0, 1, \ldots, N$, there exists a function g_n such that the price at time n is given by $V_n = g_n(X_n)$.

   (b) Suppose Y is a random variable on Ω. Define a process

$$Y_0, Y_1, \ldots, Y_N$$

by $Y_n = \widetilde{E}_n(Y)$. Let

$$Z_0, Z_1, \ldots, Z_N$$

be the Radon-Nikodym derivative process of $\widetilde{P}$ w.r.t. P, so $Z_n = E_n(Z)$, with Z the Radon-Nikodym derivative of $\widetilde{P}$ w.r.t. P. Show that the process

$$Z_0Y_0, Z_1Y_1, \ldots, Y_NZ_N$$

is a martingale w.r.t. P. (Hint: use Lemma 3.2.6)

Solution(a) The result is a direct application of Theorem 2.5.8, which can be proved with induction. Let $g_N(x) = x^2$, then $V_N = g_N(X_N)$. Assume the result is true for V_n, i.e. there exists a function g_n such that $V_n = g_n(X_n)$, we prove it is true for V_{n-1}. From the risk neutral pricing formula,

$$V_{n-1} = \widetilde{E}_{n-1}\left(\frac{V_n}{1+r}\right) = \widetilde{E}_{n-1}\left(\frac{g_n(X_n)}{1+r}\right) = \widetilde{E}_{n-1}\left(f_n(X_n)\right),$$

where $f_n(x) = \dfrac{g_n(X_n)}{1+r}$. Since the process $X_0, \ldots, X_N$ is Markov, there exists a function g_{n-1} such that

$$\widetilde{E}_{n-1}\left(f_n(X_n)\right) = g_{n-1}(X_{n-1}).$$

Thus, $V_{n-1} = g_{n-1}(X_{n-1})$ as required.

Solution(b) Using Lemma 3.2.6 with $m = N$, we get

$$Z_nY_n = Z_n\widetilde{E}_n(Y) = E_n(ZY).$$

By Theorem 3.2.1, the process $E_0(ZY)$, $E_1(ZY), \ldots, E_N(ZY)$ is a martingale w.r.t.P, therefore the process

$$Z_0Y_0, Z_1Y_1, \ldots, Y_NZ_N$$

is a martingale w.r.t. P