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Uitwerkingen Oefen Deeltentamen 2 Inleiding Financiele Wiskunde, 2011-12

1. Consider a 2-period binomial model with $S_0 = 100$, $u = 1.2$, $d = 0.7$, and $r = 0.1$. Consider now an Asian American put option with expiration $N = 2$, and intrinsic value $G_n = 95 - \dfrac{S_0 + \cdots + S_n}{n+1}$, $n = 0, 1, 2$.

   (a) Determine the price V_n at time $n = 0, 1$ of this American option.

   (b) Find the optimal exercise time $\tau^*(\omega_1\omega_2)$ for all $\omega_1\omega_2$.

   (c) Suppose it is possible to buy this option at a price $C > V_0$, where V_0 is your answer from part (a). Construct an explicit arbitrage strategy.

Solution (a): Note that the risk neutral probbaility is $\widetilde{p} = 4/5$ and $\widetilde{q} = 1/5$. The price process is given by

$$S_0 = 100, S_1(H) = 120, S_1(T) = 70, S_2(HH) = 144, S_2(HT) = S_2(TH) = 84, S@(TT) = 49.$$

The intrinsic value process is given by

$$G_0 = -5, G_1(H) = -15, G_1(T) = 10,$$

$$G_2(HH) = -26.33, G_2(HT) = -6.33, G_2(TH) = 10.33, G_2(TT) = 22.$$

The payoff at time 2 is given by

$$V_2(HH) = V_2(HT) = 0,\ V_2(TH) = 10.33,\ V_2(TT) = 22.$$

Applying the American algorithm, we get

$$V_1(H) = \max\left(-15, \frac{1}{1.1}[\frac{4}{5} \times 0 + \frac{1}{5} \times 0]\right) = 0.$$

$$V_1(T) = \max\left(10, \frac{1}{1.1}[\frac{4}{5} \times 10.33 + \frac{1}{5} \times 22]\right) = \max(10, 11.513) = 11.513.$$

$$V_0 = \max\left(-5, \frac{1}{1.1}[\frac{4}{5} \times 0 + \frac{1}{5} \times 11.513]\right) = \max(-5, 2.093) = 2.093.$$

Solution (b): The optimal exercise time is given by

$$\tau^*(HH) = \tau^*(HT) = \infty,\ \tau^*(TH) = \tau^*(TT) = 2.$$

Solution (c): Suppose it is poosible to buy the option for price $C > V_0 = 2.093$. Then at **time zero** sell the option for C, use $V_0 = 2.093$ to start a self-financing , and deposit $C - 0.293$ in the money market. To describe explicitly we first find

$$\Delta_0 = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)} = -0.23026,$$

$$\Delta_1(H) = \frac{V_2(HH) - V_2(HT)}{S_2(HH) - S_2(HT)} = 0,\ \Delta_1(T) = \frac{V_2(TH) - V_2(TT)}{S_2(TH) - S_2(TT)} = -0.33343.$$

So at time zero, the self financing portfolio has value

$$X_0 = 2.093 = \Delta_0\, S_0 + 25.119,$$

where $X_0 - \Delta_0 S_0 = 25.119$ is the money market part.

at time 1, if $\omega_1 = H$, then

$$X_1(H) = \Delta_0 S_1(H) + 1.1(25.119) = 0 = V_1(H).$$

In this case we do not need to adjust our portfolio and at time 2, $X_2(HH) = 0 = V_2(HH)$. If $\omega_1 = T$, then

$$X_1(T) = \Delta_0 S_1(T) + 1.1(25.119) = 11.513 = V_1(T).$$

If the buyer of the option decides to exercise, he gets 10, so you are left with

$$11.513 - 10 + 1.1(C - 2.093) > 0.$$

If the buyer does not exercise, then you adjust your wealth as follows

$$X_2(T) = 11.513 = \Delta_1(T) S_1(T) + 34.8531.$$

At time 2, your wealth is

$$X_2(TT) = \Delta_1(T) S_2(TT) + 1.1(34.8531) = 22,$$

which equals the payoff of the buyer. You are left with $(1.1)^2(C - 2.093) > 0$.

2. Consider the binomial model with up factor $u = 2$, down factor $d = 1/2$ and interest rate $r = 1/4$. Consider a perpetual American put option with $S_0 = 8$ and strike price $K = 10$.

   (a) Suppose the buyer of the option uses the strategy of exercising the first time the price drops to 1 euro. What is then the price at time 0 of such an option?

   (b) What is the probability that the price reaches 16 euros for the first time at time $n = 5$?

Solution (a): The buyer is using the exercise policy τ_{-3}. Hence, the price at tome 0 should be

$$\begin{aligned} V_0 = V^{\tau_{-3}} &= \widetilde{E}\left(\left(\frac{4}{5}\right)^{\tau_{-3}}(10 - S_{\tau_{-3}})\right) \\ &= (\frac{1}{2})^3(10-1) = \frac{9}{8}. \end{aligned}$$

Solution (b): The probability that the price reaches16 for the first time at time 5 is equal to the $P(\{\tau_1 = 5\})$. By Theorem5.2.5,

$$P(\{\tau_1 = 5\}) = 1/16.$$

3. Consider an American option with expiration date N, intrinsic value process $G_0, G_1, \cdots, G_N$, and price process $V_0, V_1, \cdots, V_N$. Note that

$$V_n = \max_{\tau \in \mathcal{S}_n} \widetilde{E}_n \left[\mathbf{1}_{\{\tau \leq N\}} \frac{G_\tau}{(1+r)^{\tau - n}} \right],$$

for $n = 0, 1, \cdots, N$, where r is the interest rate.

   (a) For $n = 0, 1, \cdots, N$, let $\tau_n^* \in \mathcal{S}_n$ be given by $\tau_n^* = \inf\{k \geq n : V_k = G_k\}$, if the infimum exists, otherwise $\tau_n^* = \infty$. Prove that

$$\{\frac{V_{m \wedge \tau_n^*}}{(1+r)^{m \wedge \tau_n^*}}, \ m = n, \cdots, N\}$$

   is a martingale.

   (b) Use part (a) to show τ_n^* is an optimal stopping time for V_n. i.e.

$$V_n = \widetilde{E}_n \left[\mathbf{1}_{\{\tau_n^* \leq N\}} \frac{G_{\tau_n^*}}{(1+r)^{\tau_n^* - n}} \right].$$

Solution (a): For $m \geq n$, we have

$$\frac{V_{m \wedge \tau_n^*}}{(1+r)^{m \wedge \tau_n^*}} = \frac{V_m}{(1+r)^m} \mathbb{I}_{\{\tau_n^* \geq m+1\}} + \frac{V_{\tau_n^*}}{(1+r)^{\tau_n^*}} \mathbb{I}_{\{\tau_n^* \leq m\}}.$$

Now, the random variable $\mathbb{I}_{\{\tau_n^* \geq m+1\}}$ is known at time m (since the $\{\tau_n^* \geq m+1\}^c = \{\tau_n^* \leq m\}$ is known at time m) , and on the set $\{\tau_n^* \geq m+1\}$, one has $m+1 = (m+1) \wedge \tau_n^*$, and $V_m = \widetilde{E}_m(V_{m+1}/(1+r))$. So that

$$\frac{V_m}{(1+r)^m} \mathbb{I}_{\{\tau_n^* \geq m+1\}} = \widetilde{E}_m \left(\frac{V_{(m+1) \wedge \tau_n^*}}{(1+r)^{(m+1) \wedge \tau_n^*}} \mathbb{I}_{\{\tau_n^* \geq m+1\}} \right).$$

Also, the random variable $\frac{V_{\tau_n^*}}{(1+r)^{\tau_n^*}} \mathbb{I}_{\{\tau_n^* \leq m\}}$ is known at time m, and on the set $\{\tau_n^* \leq m\}$, $\tau_n^* = (m+1) \wedge \tau_n^*$. Hence,

$$\frac{V_{\tau_n^*}}{(1+r)^{\tau_n^*}} \mathbb{I}_{\{\tau_n^* \leq m\}} = \widetilde{E}_m \left(\frac{V_{(m+1) \wedge \tau_n^*}}{(1+r)^{(m+1) \wedge \tau_n^*}} \mathbb{I}_{\{\tau_n^* \leq m\}} \right).$$

Thus,

$$\frac{V_{m\wedge\tau_n^*}}{(1+r)^{m\wedge\tau_n^*}} = E_m\left(\frac{V_{(m+1)\wedge\tau_n^*}}{(1+r)^{(m+1)\wedge\tau_n^*}}\right),$$

and therefore, $\{\frac{V_{m\wedge\tau_n^*}}{(1+r)^{m\wedge\tau_n^*}}, m = n, \cdots, N\}$ is a martingale.

Solution (b): Since $\tau_n^* \geq n$, then by part (a) we have,

$$\begin{aligned}
\frac{V_n}{(1+r)^n} = \frac{V_{n\wedge\tau_n^*}}{(1+r)^{n\wedge\tau_n^*}} &= \widetilde{E}_n\left(\frac{V_{N\wedge\tau_n^*}}{(1+r)^{N\wedge\tau_n^*}}\right) \\
&= \widetilde{E}_n\left(\frac{V_{N\wedge\tau_n^*}}{(1+r)^{N\wedge\tau_n^*}}\mathbb{I}_{\{\tau_n^*\leq N\}}\right) + \widetilde{E}_n\left(\frac{V_{N\wedge\tau_n^*}}{(1+r)^{N\wedge\tau_n^*}}\mathbb{I}_{\{\tau_n^*=\infty\}}\right) \\
&= \widetilde{E}_n\left(\frac{G_{\tau_n^*}}{(1+r)^{\tau_n^*}}\mathbb{I}_{\{\tau_n^*\leq N\}}\right),
\end{aligned}$$

where we have used that on the set $\{\tau_n^* = \infty\}$ one has $V_{N\wedge\tau_n^*} = V_N = 0$, and on the set $\{\tau_n^* \leq n\}$ on has $V_{N\wedge\tau_n^*} = V_{\tau_n^*} = G_{\tau_n^*}$.

4. Consider a 3-period (non constant interest rate) binomial model with interest rate process R_0, R_1, R_2 defined by

$$R_0 = 0,\ R_1(\omega_1) = 0.02f(\omega_1),\ R_2(\omega_1,\omega_2) = 0.02f(\omega_1)f(\omega_2)$$

where $f(H) = 3$, and $f(T) = 2$. Suppose that the risk neutral measure is given by $\widetilde{P}(HHH) = \widetilde{P}(HTT) = 1/10$, $\widetilde{P}(HHT) = \widetilde{P}(HTH) = 1/5$, $\widetilde{P}(THH) = \widetilde{P}(THT) = 1/15$, $\widetilde{P}(TTH) = \widetilde{P}(TTT) = 2/15$.

(a) Calculate the time one price $B_{1,3}$ of a zero coupon bond with maturity $m = 3$.

(b) Consider a 3-period interest rate swap. Find the 3-period swap rate SR_3, i.e. the value of K that makes the time zero no arbitrage price of the swap equal to zero.

(c) Consider a 3-period Cap that makes payments $C_n = (R_{n-1} - 0.1)^+$ at time $n = 1, 2, 3$. Find Cap_3, the price of this Cap.

Solution (a): We first calcultate the values of R_0, R_1, R_2 and D_1, D_2, D_3 in the following tables:

$\omega_1\omega_2$	R_0	R_1	R_2
HH	0	0.06	0.18
HT	0	0.06	0.12
TH	0	0.04	0.12
TT	0	0.04	0.08

$\omega_1\omega_2$	$\frac{1}{1+R_0}$	$\frac{1}{1+R_1}$	$\frac{1}{1+R_2}$	D_1	D_2	D_3	$\widetilde{P}$
HH	1	$\frac{1}{1.06}$	$\frac{1}{1.18}$	1	$\frac{1}{1.06}$	$\frac{1}{1.2508}$	$\frac{3}{10}$
HT	1	$\frac{1}{1.06}$	$\frac{1}{1.12}$	1	$\frac{1}{1.06}$	$\frac{1}{1.1872}$	$\frac{3}{10}$
TH	1	$\frac{1}{1.04}$	$\frac{1}{1.12}$	1	$\frac{1}{1.04}$	$\frac{1}{1.1648}$	$\frac{2}{15}$
TT	1	$\frac{1}{1.04}$	$\frac{1}{1.08}$	1	$\frac{1}{1.04}$	$\frac{1}{1.1232}$	$\frac{4}{15}$

Note that D_3 depends on the first two coin tosses only, and since $D_1 = 1$ we have

$$\begin{aligned} B_{1,3}(H) = \widetilde{E}_1(D_3)(H) &= D_3(HH)\widetilde{P}(\omega_2 = H|\omega_1 = H) + D_3(HT)\widetilde{P}(\omega_2 = T|\omega_1 = H) \\ &= \frac{1}{1.2508}\,\frac{1}{2} + \frac{1}{1.1872}\,\frac{1}{2} = 0.8209, \end{aligned}$$

and

$$\begin{aligned} B_{1,3}(T) = \widetilde{E}_1(D_3)(T) &= D_3(TH)\widetilde{P}(\omega_2 = H|\omega_1 = T) + D_3(TT)\widetilde{P}(\omega_2 = T|\omega_1 = T) \\ &= \frac{1}{1.1648}\,\frac{1}{3} + \frac{1}{1.1232}\,\frac{2}{3} = 0.8797. \end{aligned}$$

Solution (b): From Theorem 6.3.7, we know that

$$SR_3 = \frac{1 - B_{0,3}}{B_{0,1} + B_{0,2} + B_{0,3}}.$$

Now,

$$B_{0,1} = \widetilde{E}(D_1) = 1,$$

$$\begin{aligned} B_{0,2} = \widetilde{E}(D_2) &= \frac{1}{1.06}\,\widetilde{P}(\omega_1 = H) + \frac{1}{1.04}\,\widetilde{P}(\omega_1 = T) \\ &= \frac{1}{1.06}\,\frac{3}{5} + \frac{1}{1.04}\,\frac{2}{5} = 0.9507, \end{aligned}$$

$$\begin{aligned} B_{0,3} = \widetilde{E}(D_3) &= \frac{1}{1.2508}\,\widetilde{P}(\omega_1 = H, \omega_2 = H) + \frac{1}{1.1872}\,\widetilde{P}(\omega_1 = H, \omega_2 = T) \\ &+ \frac{1}{1.1648}\,\widetilde{P}(\omega_1 = T, \omega_2 = H) + \frac{1}{1.1232}\,\widetilde{P}(\omega_1 = H, \omega_2 = H) \\ &= \frac{1}{1.2508}\,\frac{3}{10} + \frac{1}{1.1872}\,\frac{3}{10} + \frac{1}{1.1648}\,\frac{2}{15} + \frac{1}{1.1232}\,\frac{4}{15} \\ &= 0.8444. \end{aligned}$$

Thus,

$$SR_3 = \frac{1 - B_{0,3}}{B_{0,1} + B_{0,2} + B_{0,3}} = \frac{1 - 0.8444}{2.7951} = 0.0557.$$

Solution (c): From Definition 6.3.8 we have

$$Cap_3 = \sum_{n=1}^{3} \widetilde{E}(D_n(R_{n-1} - 0.1)^+).$$

We display the values of $(R_{n-1} - 0.1)^+$ in a table

$\omega_1\omega_2$	$(R_0 - 0.1)^+$	$(R_1 - 0.1)^+$	$(R_2 - 0.1)^+$
HH	0	0	0.08
HT	0	0	0.02
TH	0	0	0.02
TT	0	0	0

Thus,

$$Cap_3 = \widetilde{E}(D_3(R_2 - 0.1)^+) = \frac{1}{1.2508}\,(0.8)\,\frac{3}{10} + \frac{1}{1.1872}\,(0.02)\,\frac{3}{10} + \frac{1}{1.1648}\,(0.02)\,\frac{2}{15} = 0.1992.$$

5. Let $M_0, M_1, \cdots$, be the symmetric random walk, i.e. $M_0 = 0$, and $M_n = \sum_{i=1}^n X_i$, where

$$X_i = \begin{cases} 1, & \text{if } \omega_i = H, \\ -1, & \text{if } \omega_i = T, \end{cases}$$

for $i \geq 1$. Let $m \geq 2$ be an integer, and let $k \in \{1, \cdots, m-1\}$. Define $Y_0 = k$, and

$$Y_{n+1} = (Y_n + X_{n+1})\mathbb{I}_{\{Y_n \notin \{0,m\}\}} + Y_n \mathbb{I}_{\{Y_n \in \{0,m\}\}},$$

for $n \geq 0$.

(a) Show that $Y_0, Y_1, \cdots$ is a martingale.

(b) Let $T = \inf\{n \geq 1 : Y_n \in \{0, m\}\}$. Using the the Optional Sampling Theorem show that $E(Y_T) = E(Y_0) = k$.

(c) Prove that $P(Y_T = 0) = \dfrac{m-k}{m}$.

Solution (a): First note that Y_n is known at time n, hence (Y_n) is an adjusted process. Since X_{n+1} is independent of the first n tosses, one has $E_n(X_{n+1}) = E(X_{n+1}) = 0$. Thus,

$$E_n(Y_{n+1}) = Y_n \mathbb{I}_{\{Y_n \notin \{0,m\}\}} + Y_n \mathbb{I}_{\{Y_n \in \{0,m\}\}} = Y_n.$$

Therefore, $Y_0, Y_1, \cdots$ is a martingale.

Solution (b): First note that $Y_{n\wedge T} = Y_n \mathbb{I}_{\{T>n\}} + Y_T \mathbb{I}_{\{T \leq n\}}$, and by the Optional Sampling Theorem, we have $(Y_{n\wedge T})$ is a martingale, so that

$$E(Y_{n\wedge T}) = E(Y_0) = k.$$

Thus, for each n,

$$\begin{aligned} Y_T &= Y_T \mathbb{I}_{\{T \leq n\}} + Y_T \mathbb{I}_{\{T>n\}} \\ &= Y_{n\wedge T} - Y_n \mathbb{I}_{\{T>n\}} + Y_T \mathbb{I}_{\{T>n\}}. \end{aligned}$$

Taking expectations gives,

$$E(Y_T) = k - E(Y_n \mathbb{I}_{\{T>n\}}) + E(Y_T \mathbb{I}_{\{T>n\}})$$

for all n. We show now that

$$\lim_{n\to\infty} E(Y_n \mathbb{I}_{\{T>n\}}) = \lim_{n\to\infty} E(Y_T \mathbb{I}_{\{T>n\}}) = 0.$$

On the set $\{T > n\}$, the random variable Y_n takes values in the set $\{1, \cdots, m-1\}$. Thus,

$$\mathbb{I}_{\{T>n\}} \leq Y_n \mathbb{I}_{\{T>n\}} \leq (m-1)\mathbb{I}_{\{T>n\}}.$$

Taking expectations gives,

$$P(\{T > n\} \leq E(Y_n \mathbb{I}_{\{T>n\}}) \leq (m-1)P(\{T > n\}).$$

Since $P(T < \infty) = 1$ (Thoerem 5.2.2), taking limits in the above inequalities gives $\lim_{n\to\infty} E(Y_n \mathbb{I}_{\{T>n\}}) = 0$. Now consider the random variable Y_T, it takes values in the set $\{0, m\}$, thus

$$0 \leq Y_T \mathbb{I}_{\{T>n\}} \leq m\mathbb{I}_{\{T>n\}},$$

so that for all n

$$0 \leq E(Y_T \mathbb{I}_{\{T>n\}} \leq mP(\{T > n\}).$$

Taking limits and using the fact that $P(T < \infty) = 1$, we get $\lim_{n\to\infty} E(Y_T \mathbb{I}_{\{T>n\}}) = 0$. This shows that $E(Y_T) = E(Y_0) = k$.

Remark (For students that have done Measure Theory): One can give a quick proof of part (b) using Lebesgue Dominated convergence Theorem and the fact that $Y_{n\wedge T}$ is bounded from below by 0 and from above by m, and that $P(T < \infty) = 1$. Using these facts, one has

$$E(Y_T) = E(\lim_{n\to\infty} Y_{n\wedge T}) = \lim_{n\to\infty} E(Y_{n\wedge T}) = E(Y_0) = k.$$

Solution (c): Note that Y_T takes only two values 0 and m. Let $p = P(Y_T = 0)$, then $P(Y_T = m) = 1 - p$, and $E(Y_T) = m(1 - p)$. On the other hand, by part (b) we have $E(Y_T) = k$, thus $m(1 - p) = k$ implying that $P(Y_T = 0) = p = \dfrac{m-k}{m}$.