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Measure and Integration Exercises 2

1. Let $a < s < b$, and suppose $f : [a,b] \to \mathbb{R}$ is bounded and continuous at s. Let $\Psi : [a,b] \to R$ be given by

$$\Psi(x) = \begin{cases} 0 & \text{if } a \leq x \leq s \\ 1 & \text{if } s < x \leq b. \end{cases}$$

Show that f is Ψ-Riemann integrable, and $\int_a^b f(x)d\Psi(x) = f(s)$.

2. Let $a = a_0 < a_1 < a_2 < \cdots < a_n = b$, and suppose that the function $\Psi : [a,b] \to R$ has the constant value c_i on the interval (a_{i-1}, a_i) for $i = 1, 2, \cdots, n$. Show that if $f : [a,b] \to \mathbb{R}$ is continuous, then f is Ψ-Riemann integrable, and

$$\int_a^b f(x)d\Psi(x) = \sum_{i=0}^{n} f(a_i)d_i,$$

where

$$d_i = \begin{cases} c_1 - \Psi(a) & \text{if } i = 0 \\ c_{i+1} - c_i & \text{if } 1 \leq i \leq n-1 \\ \Psi(b) - c_n & \text{if } i = n. \end{cases}$$

3. Let $\Psi : [a,b] \to R$ be non-decreasing, and let $f : [a,b] \to \mathbb{R}$ be bounded. Show that f is Ψ-Riemann integrable **if and only if** for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\sum_{\{I \in \mathcal{C} \,:\, \sup_{x \in I} f(x) - \inf_{x \in I} f(x) \geq \epsilon\}} \Delta_I \Psi < \epsilon$$

for all finite non-overlapping exact covers $\mathcal{C}$ of $[a,b]$ such that $||\mathcal{C}|| < \delta$.

4. Let $\Psi : [a,b] \to R$ be non-decreasing, and $f : [a,b] \to \mathbb{R}$ be bounded. Show that if f is Ψ-Riemann integrable, then the function $f^2; [a,b] \to \mathbb{R}$ given by $f^2(x) = (f(x))^2$ is Ψ-Riemann integrable.