Universiteit Utrecht
Mathematisch Instituut

Universiteit Utrecht

Boedapestlaan 6
3584 CD Utrecht

Measure and Integration Exercises 3

1. Suppose $\Psi : [a, b] \to \mathbb{R}$ is non-decreasing, and $f : [a, b] \to \mathbb{R}$ is Ψ-Riemann integrable. Assume that $m < M$, and $m \leq f(x) \leq M$. Let $g : [m, M] \to \mathbb{R}$ be continuous. Show that the function $g \circ f : [a, b] \to \mathbb{R}$ is Ψ-Riemann integrable.

2. Let $A, B \subseteq \mathbb{R}^N$. Prove the following.

   (a) If $|A|_e = 0$, then $|A \cup B|_e = |B|_e$.

   (b) If $|A \Delta B|_e = 0$, then $|A \cup B|_e = |A|_e = |B|_e = |A \cap B|_e$.

3. Let K_1, K_2 be compact subsets of $\mathbb{R}^n$ such that $K_1 \cap K_2 = \emptyset$. Show that

$$|K_1 \cup K_2|_e = |K_1|_e + |K_2|_e.$$

4. Let F be a closed subset of $\mathbb{R}^N$. For each $n \geq 1$, let

$$G_n = \{x \in \mathbb{R}^N : |x - y| < \frac{1}{n} \text{ for some } y \in F\}.$$

   (a) Show that G_n is open for each $n \geq 1$.

   (b) Show that $F = \bigcap_{n=1}^{\infty} G_n$.

   (c) Conclude that $\mathcal{F} \subseteq \mathcal{O}_\delta$. Here $\mathcal{F}$ denotes the collection of all closed subset of $\mathbb{R}^N$, and $\mathcal{O}_\delta$ denotes the collections of all subsets of $\mathbb{R}^N$ that can be written as the countable intersection of open setsets of $\mathbb{R}^N$.