Universiteit Utrecht
Mathematisch Instituut

Universiteit Utrecht

Boedapestlaan 6
3584 CD Utrecht

# Measure and Integration Exercises 6

1. Let $E$ be an uncountable set, and $\mathcal{B} = \{A \subseteq E : A \text{ or } A^c \text{ is countable}\}$. Show that $\mathcal{B}$ is a $\sigma$-algebra over $E$.

2. Let $E$ be a set, and $\mathcal{C} \subseteq \mathcal{P}(E)$. Consider $\sigma(E;\mathcal{C})$, the smallest $\sigma$-algebra over $E$ containing $\mathcal{C}$, and let $\mathcal{D}$ be the collection of sets $A \in \sigma(E;\mathcal{C})$ with the property that there exists a countable collection $\mathcal{C}_0 \subseteq \mathcal{C}$ (depending on $A$) such that $A \in \sigma(E;\mathcal{C}_0)$.

   (a) Show that $\mathcal{D}$ is a $\sigma$-algebra over $E$.

   (b) Show that $\mathcal{D} = \sigma(E;\mathcal{C})$.

3. A collection $\mathcal{M}$ of sets is said to be a *monotone class* if it satisfies the following two properties:

   (i) if $\{A_n\} \subseteq \mathcal{M}$ with $A_1 \subseteq A_2 \subseteq \ldots$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{M}$, and

   (ii) if $\{B_n\} \subseteq \mathcal{M}$ with $B_1 \supseteq B_2 \supseteq \ldots$, then $\bigcap_{n=1}^{\infty} B_n \in \mathcal{M}$.

   (a) Show that the intersection of an arbitrary collection of monotone classes is a monotone class.

   (b) Let $E$ be a set, and $\mathcal{B}$ a collection of subsets of $E$. Show that $\mathcal{B}$ is a $\sigma$-algebra if and only if $\mathcal{B}$ is an algebra and a monotone class.

   (c) Let $\mathcal{A}$ be an algebra over $E$, and $\mathcal{M}$ the smallest monotone class containing $\mathcal{A}$, i.e. $\mathcal{M}$ is the intersection of all monotone classes containing $\mathcal{A}$. Show that $\mathcal{M}$ is an algebra.

   (d) Using the same notation as in part (c), show that $\mathcal{M} = \sigma(E, \mathcal{A})$, where $\sigma(E, \mathcal{A})$ is the smallest $\sigma$-algebra over $E$ containg the algebra $\mathcal{A}$.