Universiteit Utrecht
Mathematisch Instituut

Universiteit Utrecht

Boedapestlaan 6
3584 CD Utrecht

Measure and Integration Exercises 10

1. Let $(E, \mathcal{B}, \mu)$ be a measure space. Let (f_n) be a sequence of non-negative measurable functions.

   (a) Prove that
   $$\int_E \sum_{n=1}^{\infty} f_n \, d\mu = \sum_{n=1}^{\infty} \int_E f_n \, d\mu.$$

   (b) Let (g_n) be a sequence of μ-integrable functions on E such that $\sum_{n=1}^{\infty} \int_E |g_n| \, d\mu < \infty$. Show that $\sum_{n=1}^{\infty} g_n$ is finite μ almost everywhere, and
   $$\int_E \sum_{n=1}^{\infty} g_n \, d\mu = \sum_{n=1}^{\infty} \int_E g_n \, d\mu.$$

   (c) Let f be a non-negative integrable function on E. Define ν on $\mathcal{B}$ by
   $$\nu(A) = \int_A f \, d\mu.$$
   Show that ν is a finite measure on $\mathcal{B}$.

2. Consider the measure space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$, where μ is the counting measure on $\mathcal{P}(\mathbb{N})$, i.e. $\mu(A)$ is equal to the number of elements in A.

   (a) Show that for any $f : \mathbb{N} \to [0, \infty]$, one has
   $$\int_{\mathbb{N}} f \, d\mu = \sum_{k=1}^{\infty} f(k).$$

   (b) For each $n \geq 1$, let $(a_k^n)_k$ be a sequence of real numbers such that $0 \leq a_k^n \leq a_k^{n+1}$ for all k and n. Show that
   $$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_k^n = \sum_{k=1}^{\infty} \lim_{n\to\infty} a_k^n.$$

3. Let $(E, \mathcal{B}, \mu)$ be a measure space, and $f : E \to [0, \infty]$ a measurable function.

   (a) Show that if $\int_E f \, d\mu < \infty$, then $\lim_{n\to\infty} n\mu(f \geq n) = 0$.

   (b) Suppose that $\mu(E) < \infty$. Show that
   $$\int_E f \, d\mu < \infty \textbf{ if and only if } \sum_{n=0}^{\infty} \mu(f > n) < \infty.$$