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Measure and Integration Exercises 11

1. Let $(E, \mathcal{B}, \mu)$ be a measure space, and $f_n : E \to \mathbb{R}$ a sequence of measurable real valued functions on $(E, \mathcal{B}, \mu)$.

   (a) Suppose $f : E \to \mathbb{R}$ is measurable. Show that
   $$\{x \in E : \lim_{n\to\infty} f_n(x) \neq f(x)\} = \bigcup_{l=1}^{\infty} \bigcap_{m=1}^{\infty} \{x \in E : \sup_{n\geq m} |f_n(x) - f(x)| \geq 1/l\}.$$

   (b) Show that if $f_n \to f$ μ a.e., then for every $\epsilon > 0$
   $$\mu(\bigcap_{m=1}^{\infty} \{x \in E : \sup_{n\geq m} |f_n(x) - f(x)| \geq \epsilon\}) = 0.$$

2. Consider the measure space $([0,1), \mathcal{B}_{[0,1)}, \lambda_{[0,1)})$, where $\mathcal{B}_{[0,1)}$ and $\lambda_{[0,1)}$ are the restrictions of the Borel σ-algebra and Lebesgue measure on $[0,1)$. Define a sequence of measurable functions f_n on $[0,1)$ as follows: given $n \geq 1$, there exist an $m \geq 0$ and $0 \leq l \leq 2^m - 1$ such that $n = 2^m + l$ (note that this representation is unique). Set $f_n = f_{2^m+l} = 1_{[l/2^m,(l+1)/2^m]}$.

   (a) Determine explicitly $f_1, f_2, f_3, f_4, f_5, f_6, f_7$.

   (b) Show that $\limsup_{n\to\infty} f_n(x) = 1$ for all $x \in [0,1)$.

   (c) Show that $\lim_{n\to\infty} ||f||_{L^1(\lambda_{[0,1)})} = 0$. Conclude that L^1-convergence does not imply μ a.e. convergence.

3. Consider the measure space $([a,b], \mathcal{B}, \lambda)$, where $\mathcal{B}$ is the Borel σ-algebra on $[a,b]$, and λ is the restriction of the Lebesgue measure on $[a,b]$. Let $f : [a,b] \to \mathbb{R}$ be any continuous function. Show that the Riemann integral of f on $[a,b]$ is equal to the Lebesgue integral of f on $[a,b]$, i.e.
   $$(R)\int_a^b f(x)dx = \int_{[a,b]} f d\lambda.$$

4. Let $(E, \mathcal{B}, \mu)$ be a measure space, and $f_n : E \to \mathbb{R}$ a sequence of measurable real valued functions on $(E, \mathcal{B}, \mu)$. Let $f : E \to \mathbb{R}$ be a measurable function such that $\sum_{n=0}^{\infty} \mu(|f - f_n| \geq \epsilon)) < \infty$ for all $\epsilon > 0$. Show that $f_n \to f$ in μ-measure and μ a.e.