Measure and Integration Exercises 12

1. Let $(E, \mathcal{B}, \mu)$ be a measure space, and $f_n : E \to \mathbb{R}$ a sequence of measurable real valued functions on $(E, \mathcal{B}, \mu)$. Suppose $f, g : E \to \mathbb{R}$ are measurable functions such that $f_n \to f$ in μ-measure and $f_n \to g$ μ a.e. Show that $f = g$ μ a.e.

2. Consider the measure space $([0, \infty), \mathcal{B}, \lambda)$, where $\mathcal{B}$ and λ are the restriction of the Borel σ-algebra and Lebesgue measure to the interval $[0, \infty)$. Define $f_n : [0, \infty) \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} 1 & \text{if } n \leq x \leq n + \frac{1}{n} \\ \\ 0 & \text{elsewhere .} \end{cases}$$

   (a) Prove that $f_n \to 0$ λ a.e. and in λ-measure.

   (b) Prove that condition (3.3.8) of Theorem 3.3.7 does not hold, i.e. **it is not true** that
$$\lim_{m\to\infty} \lambda(\sup_{n\geq m} |f_n| \geq \epsilon) = 0 \ \text{ for all } \epsilon > 0.$$

3. Let $(E, \mathcal{B}, \mu)$ be a measure space, and $f_n : E \to \mathbb{R}$ a sequence of measurable real valued functions on $(E, \mathcal{B}, \mu)$. Let (ϵ_n) be a sequence of positive real numbers such that $\sum_n \epsilon_n < \infty$. Prove that if $\sum_{n=0}^{\infty} \mu(|f_{n+1} - f_n| \geq \epsilon_n)) < \infty$, then there exists a measurable function $g : E \to R$ such that $f_n \to g$ in μ-measure and μ a.e.

4. Let f and $\{f_n\}$ be measurable real valued functions on a measure space $(E, \mathcal{B}, \mu)$ such that $f_n \to f$ in μ-measure, and $\sup_{n\geq 1} ||f_n||_{L^1(\mu)} < \infty$. Show that f is μ-integrable, and

$$\lim_{n\to\infty} \left| ||f_n||_{L^1(\mu)} - ||f||_{L^1(\mu)} - ||f_n - f||_{L^1(\mu)} \right| = || \, |f_n| - |f| - |f_n - f| \, ||_{L^1(\mu)} = 0.$$

   Conclude that if $||f_n||_{L^1(\mu)} \to ||f||_{L^1(\mu)} \in \mathbb{R}$, then $||f_n - f||_{L^1(\mu)} \to 0$.