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Measure and Integration Exercises 13

1. Let $(L, (\ ,\ ))$ be an inner product space, and let $||x||_L = (x, x)^{1/2}$. $x \in L$.
   (a) Let $(x_n) \subseteq L$, and $x \in L$. Show that if $\lim\limits_{n\to\infty} ||x_n - x||_L = 0$, then $\lim\limits_{n\to\infty} ||x_n||_L = ||x||_L$.
   (b) Prove that the inner product $(\ ,\ )$ is jointly continuous, i.e. if $\lim\limits_{n\to\infty} ||x_n - x||_L = 0$ and $\lim\limits_{n\to\infty} ||y_n - y||_L = 0$, then $\lim\limits_{n\to\infty} (x_n, y_n) = (x, y)$.

2. Let $(E, \mathcal{B}, \mu)$ be a measure space, and let $\{f_n\} \subseteq L^2(\mu)$ be such that

$$\lim_{m\to\infty} \sup_{n\geq m} ||f_n - f_m||_{L^2(\mu)} = 0.$$

   Show that there exists a function $f \in L^2(\mu)$ such that $\lim\limits_{n\to\infty} ||f_n - f||_{L^2(\mu)} = 0$. In other words $(L^2(\mu), ||\ ||_{L^2(\mu)})$ is a complete metric space.

3. Let $(E, \mathcal{B}, \mu)$ be a finite measure space. Show that $L^2(\mu) \subseteq L^1(\mu)$. Show that the result is not true in case μ is not a finite measure

4. Let μ and ν be two measures on the measure space $(E, \mathcal{B})$ such that $\mu(A) \leq \nu(A)$ for all $A \in \mathcal{B}$. Show that if f is any non-negative measurable function on $(E, \mathcal{B})$, then $\int_E f\, d\mu \leq \int_E f\, d\nu$. Conclude that if ν is a finite measure, then $L^2(\nu) \subseteq L^1(\nu) \subseteq L^1(\mu)$.