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Measure and Integration Exercises 13

1. Let $(E_1, \mathcal{B}_1, \mu_1)$ and $(E_2, \mathcal{B}_2, \mu_2)$ be finite measure spaces. Let $\Gamma \in \mathcal{B}_1 \times \mathcal{B}_2$. For $x_1 \in E_1$, $x_2 \in E_2$, let $\Gamma(x_1) = \{x_2 \in E_2 : (x_1, x_2) \in \Gamma\}$ and $\Gamma(x_2) = \{x_1 \in E_1 : (x_1, x_2) \in \Gamma\}$. Let

$$\mathcal{D} = \{\Gamma \in \mathcal{B}_1 \times \mathcal{B}_2 : \mu_2(\Gamma(x_1)) \text{ and } \mu_1(\Gamma(x_2)) \text{ are measurable }\}.$$

Show that $\mathcal{D}$ is a λ-system containing $\mathcal{C} = \{\Gamma_1 \times \Gamma_2 : \Gamma_i \in \mathcal{B}_i, i = 1, 2\}$. Conclude that $\mathcal{D} = \mathcal{B}_1 \times \mathcal{B}_2$.

2. Suppsose that μ is σ-finite on $(E, \mathcal{B})$, and write $E = \bigcup_{n=1}^{\infty} E_n$, where E_n are measurable, pairwise disjoint and $\mu(E_n) < \infty$. Define μ_n on $\mathcal{B}$ by $\mu_n(A) = \mu(A \cap E_n)$. Show that μ_n is a measure on $(E, \mathcal{B})$ and for every $f \geq 0$ measurable,

$$\int_E f \, d\mu = \sum_{n=1}^{\infty} \int_{E_n} f \, d\mu_n.$$

3. Let $(E_1, \mathcal{B}_1, \mu_1)$ and $(E_2, \mathcal{B}_2, \mu_2)$ be σ-finite measure spaces. Let $\Gamma \in \mathcal{B}_1 \times \mathcal{B}_2$. For $x_1 \in E_1$, $x_2 \in E_2$, let $\Gamma(x_1) = \{x_2 \in E_2 : (x_1, x_2) \in \Gamma\}$ and $\Gamma(x_2) = \{x_1 \in E_1 : (x_1, x_2) \in \Gamma\}$. Show that the following are equivalent:

   (i) $(\mu_1 \times \mu_2)(\Gamma) = 0$,

   (ii) $\mu_1(\Gamma(x_2)) = 0$ for μ_2 almost every $x_2 \in E_2$,

   (iii) $\mu_2(\Gamma(x_1)) = 0$ for μ_1 almost every $x_1 \in E_1$.

4. Consider $(\mathbb{R}, \mathcal{B}, \lambda)$, where $\mathcal{B}$ is the Borel σ-algebra, λ is Lebesgue measure and μ is counting measure (i.e. $\mu(B)$ = number of elements in B). Let $A = \{(x, y) : x = y\}$, show that

$$\int_{\mathbb{R}} \int_{\mathbb{R}} 1_A(x_1, x_2) d\lambda(x_1) d\mu(x_2) = 0$$

while

$$\int_{\mathbb{R}} \int_{\mathbb{R}} 1_A(x_1, x_2) d\mu(x_2) d\lambda(x_1) = \infty.$$

Why does not this violate Tonelli's Theorem?