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Measure and Integration Solutions 14

1. Let $(E, \mathcal{B})$ be a measure space, and μ_1, μ_2 and λ σ-finite measures on $(E, \mathcal{B})$.
   (a) If $\mu_1 \perp \nu$ and $\mu_2 \perp \nu$, then $\mu_1 + \mu_2 \perp \nu$.
   (b) If $\mu_1 \ll \nu$ and $\mu_2 \perp \nu$, then $\mu_1 \perp \mu_2$.
   (c) If $\mu_1 \ll \nu$ and $\mu_1 \perp \nu$, then μ_1 is the zero measure.

2. Suppose μ is a finite measure and ν a σ-finite measure $(E, \mathcal{B})$. Show that the Lebesgue decomposition of μ with respect to ν is unique, i.e. prove that if $\mu = \mu_a + \mu_\sigma = \mu'_a + \mu'_\sigma$ with $\mu_a \ll \nu$, $\mu'_a \ll \nu$, $\mu_\sigma \perp \nu$ and $\mu'_\sigma \perp \nu$, then $\mu_a = \mu'_a$ and $\mu_\sigma = \mu'_\sigma$.

3. Consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra. Define σ on $\mathcal{B}(\mathbb{R})$ by $\sigma(\Gamma) = \sum_{n \in \mathbb{Z} \cap \Gamma} \frac{1}{n^2}$.
   (a) Show that σ is a measure on $\mathcal{B}(\mathbb{R})$ such that $\sigma \perp \lambda$, where λ is Lebesgue measure on $\mathcal{B}(\mathbb{R})$.
   (b) Let $f \in L^1(\lambda)$ be non-negative, and define μ on $\mathcal{B}(\mathbb{R})$ by $\mu(\Gamma) = \int_\Gamma f\, d\lambda$. Let $\nu = \mu + \sigma$. Find the Lebesgue decomposition of ν with respect to λ.

4. Let $(E, \mathcal{B}, \nu)$ be a measure space, and $h : E \to \mathbb{R}$ a non-negative measurable function. Define a measure μ on $(E, \mathcal{B})$ by $\mu(A) = \int_A h d\nu$ for $A \in \mathcal{B}$. Show that for every measurable function $F : E \to \mathbb{R}$ one has
$$\int_E F\, d\mu = \int_E F h\, d\nu$$
in the sense that if one integral exists, then the other integral also exists, and they are equal.

5. Suppose that μ and ν are finite measures on $(E, \mathcal{B})$ such that $\mu \ll \nu$ and $\nu \ll \mu$, i.e. μ and ν have the same sets of measure zero. Show that the Radon-Nikodym derivatives $\frac{d\mu}{d\nu}$ and $\frac{d\nu}{d\mu}$ are positive ν a.e. (and hence μ a.e.) and $\frac{d\mu}{d\nu} \cdot \frac{d\nu}{d\mu} = 1$ ν and μ a.e.