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Measure and Integration Exercises 7

1. Let E be a set, and $\mathcal{C} \subseteq \mathcal{P}(E)$. Consider $\sigma(E;\mathcal{C})$, the smallest σ-algebra over E containing $\mathcal{C}$, and let $\mathcal{D}$ be the collection of sets $A \in \sigma(E;\mathcal{C})$ with the property that there exists a countable collection $\mathcal{C}_0 \subseteq \mathcal{C}$ (depending on A) such that $A \in \sigma(E;\mathcal{C}_0)$.

   (a) Show that $\mathcal{D}$ is a σ-algebra over E.

   (b) Show that $\mathcal{D} = \sigma(E;\mathcal{C})$.

2. Let $M \subset \mathbb{R}$ be a non-Lebesgue measurable set (i.e. $M \notin \overline{\mathcal{B}}_{\mathbb{R}}$.). Define $A = \{(x,x) \in \mathbb{R}^2 : x \in M\}$, and let $g : \mathbb{R} \to \mathbb{R}^2$ be given by $g(x) = (x,x)$.

   (a) Show that $A \in \overline{\mathcal{B}}_{\mathbb{R}^2}$. i.e. A is Lebesgue measurable. (Hint: use the fact that Lebesgue measure is rotation invariant).

   (b) Show that g is a Borel-measurable function, i.e. $g^{-1}(B) \in \mathcal{B}_{\mathbb{R}}$ for each $B \in \mathcal{B}_{\mathbb{R}^2}$.

   (c) Show that $A \notin \mathcal{B}_{\mathbb{R}^2}$, i.e. A is not Borel measurable.

3. Let $(E, \mathcal{B}, \mu)$ be a measure space, and $\{A_n\}$ a sequence in $\mathcal{B}$. Define

$$\limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m,$$

   and

$$\liminf_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} A_m.$$

   (a) Prove that $\mu(\liminf_{n\to\infty} A_n) \le \liminf_{n\to\infty} \mu(A_n)$.

   (b) Suppose that $\mu(\bigcup_{n=1}^{\infty} A_n) < \infty$. Prove that $\mu(\limsup_{n\to\infty} A_n) \ge \limsup_{n\to\infty} \mu(A_n)$.

   (c) Prove that if $\sum_{n=1}^{\infty} \mu(A_n) < \infty$, then $\mu(\limsup_{n\to\infty} A_n) = 0$. (This is known as the Borel-Cantelli Lemma).

4. Let $\mathcal{C} = \{(a,\infty) : a \in \mathbb{R}\}$, and let $\mathcal{B}_{\mathbb{R}}$ be the Borel σ-algebra over $\mathbb{R}$.

   (a) Show that $\mathcal{B}_{\mathbb{R}} = \sigma(E;\mathcal{C})$.

   (b) Let $(E, \mathcal{B}, \mu)$ be a **finite** measure space. Suppose $f : E \to \mathbb{R}$ satisfies $f^{-1}((a,\infty)) \in \mathcal{B}$ for all $a \in \mathbb{R}$. Show that f is measurable, i.e. $f^{-1}(A) \in \mathcal{B}$ for all $A \in \mathcal{B}_{\mathbb{R}}$.

   (c) Suppose ν is a finite measure on $\mathcal{B}_{\mathbb{R}}$, and $\mu\left(f^{-1}(a,\infty)\right) = \nu\left((a,\infty)\right)$ for all $a \in \mathbb{R}$. Show that $\mu(f^{-1}(A)) = \nu(A)$ for all $A \in \mathcal{B}_{\mathbb{R}}$.