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Measure and Integration Solutions 7

1. Suppose E is a set, $\mathcal{C}$ a π-system over E and $\mathcal{B} = \sigma(E;\mathcal{C})$ (the smallest σ-algebra over E containing $\mathcal{C}$). Let μ and ν be two measures on $(E, \mathcal{B})$ such that (i) $\mu(E) = \nu(E) < \infty$, and (ii) $\mu(C) = \nu(C)$ for all $C \in \mathcal{C}$. Let $\mathcal{H} = \{A \in \mathcal{B} : \mu(A) = \nu(A)\}$.
   (a) Show that $\mathcal{H}$ is a λ-system over E.
   (b) Show that $\mathcal{B} = \mathcal{H}$, and conclude that $\mu(A) = \nu(A)$ for all $A \in \mathcal{B}$.

2. Let $(E, \mathcal{B}, \mu)$ be a measure space, and $\overline{\mathcal{B}}^{\mu}$ be the completion of the σ-algebra $\mathcal{B}$ with respect to the measure μ. We denote by $\overline{\mu}$ the extension of the measure μ to the σ-algebra $\overline{\mathcal{B}}^{\mu}$. Suppose $f : E \to E$ is a function such that $f^{-1}(B) \in \mathcal{B}$ and $\mu(f^{-1}(B)) = \mu(B)$ for each $B \in \mathcal{B}$, where $f^{-1}(B) = \{x \in E : f(x) \in B\}$. Show that $f^{-1}(\Gamma) \in \overline{\mathcal{B}}^{\mu}$ and $\overline{\mu}(f^{-1}(\Gamma)) = \overline{\mu}(\Gamma)$ for all $\Gamma \in \overline{\mathcal{B}}^{\mu}$.

3. Let $(E, \mathcal{B}, \mu)$ be a measure space, and $\{A_n\}$ a sequence in $\mathcal{B}$. Define

$$\limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m,$$

and

$$\liminf_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} A_m.$$

   (a) Prove that $\mu(\liminf_{n\to\infty} A_n) \leq \liminf_{n\to\infty} \mu(A_n)$.
   (b) Suppose that $\mu(\bigcup_{n=1}^{\infty} A_n) < \infty$. Prove that $\mu(\limsup_{n\to\infty} A_n) \geq \limsup_{n\to\infty} \mu(A_n)$.
   (c) Prove that if $\sum_{n=1}^{\infty} \mu(A_n) < \infty$, then $\mu(\limsup_{n\to\infty} A_n) = 0$. (This is known as the Borel-Cantelli Lemma).

4. Let $\mathcal{C} = \{(a, \infty) : a \in \mathbb{R}\}$, and let $\mathcal{B}_{\mathbb{R}}$ be the Borel σ-algebra over $\mathbb{R}$.
   (a) Show that $\mathcal{B}_{\mathbb{R}} = \sigma(E;\mathcal{C})$.
   (b) Let $(E, \mathcal{F}, \mu)$ be a **finite** measure space. Suppose $f : E \to \mathbb{R}$ satisfies $f^{-1}(A) \in \mathcal{F}$ for all $A \in \mathcal{B}_{\mathbb{R}}$, where $\mathcal{B}_{\mathbb{R}}$ is the Borel σ-algebra over $\mathbb{R}$. Define μ_f on $\mathcal{B}_{\mathbb{R}}$ by $\mu_f(A) = \mu(f^{-1}(A))$ for all $A \in \mathcal{B}_{\mathbb{R}}$. Show that μ_f is a measure on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$.