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# Measure and Integration Exercises 8

1. Let $(E, \mathcal{B}, \mu)$ be a measure space, and $\overline{\mathcal{B}}^{\mu}$ be the completion of the $\sigma$-algebra $\mathcal{B}$ with respect to the measure $\mu$. We denote by $\overline{\mu}$ the extension of the measure $\mu$ to the $\sigma$-algebra $\overline{\mathcal{B}}^{\mu}$. Suppose $f : E \to E$ is a function such that $f^{-1}(B) \in \mathcal{B}$ and $\mu(f^{-1}(B)) = \mu(B)$ for each $B \in \mathcal{B}$. Show that $f^{-1}(\Gamma) \in \overline{\mathcal{B}}^{\mu}$ and $\overline{\mu}(f^{-1}(\Gamma)) = \overline{\mu}(\Gamma)$ for all $\Gamma \in \overline{\mathcal{B}}^{\mu}$.

2. Let $(E, \mathcal{B}, \mu)$ be a measure space, and $f_n : E \to [-\infty, \infty]$ a sequence of measurable functions. Show that $\sup_n f_n$ and $\inf_n f_n$ are measurable.

3. Let $(E, \mathcal{B}, \mu)$ be a measure space. Suppose $f : E \to [-\infty, \infty]$ is a function such that $f = \sum_{i=1}^{n} a_i 1_{A_i}$, where $a_1, \cdots, a_n$ are **distinct** elements of $[-\infty, \infty]$ and $A_1, A_2, \cdots, A_n$ are disjoint subsets of $E$. Show that $f$ is measurable (i.e. $f^{-1}(A) \in \mathcal{B}$ for all $A \in \mathcal{B}_{[-\infty,\infty]}$) **if and only if** $A_1, A_2, \cdots, A_n \in \mathcal{B}$.

4. Let $(E, \mathcal{B}, \mu)$ be a measure space, and $f : E \to [0, \infty]$ a measurable simple function such that $\int_E f \, d\mu < \infty$. Define $\lambda : \mathcal{B} \to [0, \infty]$ by

$$\lambda(B) = \int_B f \, d\mu.$$

   (a) Show that $\lambda$ is a **finite** measure on $\mathcal{B}$.

   (b) Suppose that $\mu(f = 0) = 0$. Show that $\lambda(B) = 0$ **if and only if** $\mu(B) = 0$.