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Measure and Integration: Extra Exercises

1. Let $(E, \mathcal{B}, \mu)$ be a probability space, i.e. $\mu(E) = 1$. Let $f : E \to [0, 1)$ be a measurable function such that $\mu\left(f^{-1}([\frac{k}{2^n}, \frac{k+1}{2^n})\right) = \frac{1}{2^n}$ for $n \geq 1$ and $k = 0, 1, \cdots, 2^n - 1$. Show that $\int_E f^2 \, d\mu = \frac{1}{3}$.

   Proof Let $A_{k,n} = f^{-1}([\frac{k}{2^n}, \frac{k+1}{2^n}))$, for $n \geq 1$ and $k = 0, 1, \cdots, 2^n - 1$. For $n \geq 1$, let $g_n = \sum_{k=0}^{2^n-1} \frac{k^2}{4^n} 1_{A_{k,n}}$. Then, g_n is a sequence of non-negative measurable functions such that $g_n \uparrow f^2$. Furthermore,

   $$\int_E g_n \, d\mu = \sum_{k=0}^{2^n-1} \frac{k^2}{8^n} = \frac{2^n(2^n-1)(2^{n+1}+1)}{6 \cdot 8^n}.$$

   By the Monotone Convergence Theorem,

   $$\int_E f^2 \, d\mu = \lim_{n\to\infty} \int_E g_n \, d\mu = \lim_{n\to\infty} \frac{2^n(2^n-1)(2^{n+1}+1)}{6 \cdot 8^n} = \frac{1}{3}.$$

2. Consider the measure space $([a, b], \mathcal{B}, \lambda)$, where $\mathcal{B}$ is the Borel σ-algebra on $[a, b]$, andλ is the restriction of the Lebesgue measure on $[a, b]$. Let $f : [a, b] \to \mathbb{R}$ be a bounded Riemann integrable function. Show that the Riemann integral of f on $[a, b]$ is equal to the Lebesgue integral of f on $[a, b]$, i.e.

   $$(R) \int_a^b f(x)dx = \int_{[a,b]} f d\lambda.$$

   Proof For each $n \geq 1$, divide the interval $[a, b]$ into 2^n intervals of equal length $I_0^{(n)}, I_1^{(n)}, \cdots, I_{2^n-1}^{(n)}$, where

   $$I_j^{(n)} = \left[a + \frac{j(b-a)}{2^n}, a + \frac{(j+1)(b-a)}{2^n}\right].$$

   Let $\mathcal{C}^{(n)} = \{I_j^{(n)} : 0 \leq j \leq 2^n - 1.\}$. Notice that $\mathcal{C}^{(n+1)}$ is a refinement of $\mathcal{C}^{(n)}$, $||\mathcal{C}^{(n)}|| = \frac{1}{2^n} \to 0$ as $n \to \infty$ and

   $$\lim_{n\to\infty} \mathcal{U}(f; \mathcal{C}^{(n)}) = (R) \int_a^b f(x)dx = \lim_{n\to\infty} \mathcal{L}(f; \mathcal{C}^{(n)}),$$

where $\mathcal{U}, \mathcal{L}$ denote the upper and lower Riemann sums respectively. For each $n \geq 1$ and $0 \leq j \leq 2^n - 1$, let

$$M_j^{(n)} = \sup_{x \in I_j^{(n)}} f(x), \text{and } m_j^{(n)} = \inf_{x \in I_j^{(n)}} f(x).$$

Define for $n \geq 1$,

$$f_n(x) = \begin{cases} f(a) & \text{if } x = a \\ M_j^{(n)} & \text{if } x \in \left(a + \frac{j(b-a)}{2^n}, a + \frac{(j+1)(b-a)}{2^n}\right], 0 \leq j \leq 2^n - 1, \end{cases}$$

and

$$g_n(x) = \begin{cases} f(a) & \text{if } x = a \\ m_j^{(n)} & \text{if } x \in \left(a + \frac{j(b-a)}{2^n}, a + \frac{(j+1)(b-a)}{2^n}\right], 0 \leq j \leq 2^n - 1. \end{cases}$$

Then,

$$\int_{[a,b]} g_n d\lambda = \mathcal{L}(f; \mathcal{C}^{(n)}) \text{ and } \int_{[a,b]} f_n d\lambda = \mathcal{U}(f; \mathcal{C}^{(n)}),$$

and

$$g_1 \leq g_2 \leq \cdots \leq f \leq \cdots \leq f_2 \leq f_1.$$

Since $\{g_n\}$ is a bounded increasing sequence, and $\{f_n\}$ is a bounded decreasing sequence, there exist measurable functions F and G such that

$$G = \lim_{n \to \infty} g_n \text{ and } F = \lim_{n \to \infty} f_n.$$

Furthermore, $g_1 \leq G \leq f \leq F \leq f_1$ and hence

$$\int_{[a,b]} G d\lambda = \int_{[a,b]} f d\lambda = \int_{[a,b]} F d\lambda.$$

By the Lebesgue Dominated Convergence Theorem,

$$\int_{[a,b]} G d\lambda = \lim_{n \to \infty} \int_{[a,b]} g_n d\lambda = \lim_{n \to \infty} \mathcal{L}(f; \mathcal{C}^{(n)}) = (R) \int_a^b f(x) dx$$

and

$$\int_{[a,b]} F d\lambda = \lim_{n \to \infty} \int_{[a,b]} f_n d\lambda = \lim_{n \to \infty} \mathcal{U}(f; \mathcal{C}^{(n)}) = (R) \int_a^b f(x) dx.$$

Thus,

$$\int_{[a,b]} f d\lambda = \int_{[a,b]} G d\lambda = \int_{[a,b]} F d\lambda = (R) \int_a^b f(x) dx.$$

3. Let $0 < a < b$. Prove with the help of Fubini's theorem that $\int_0^\infty (e^{-at} - e^{-bt}) \frac{1}{t} dt = \log(b/a)$.

Proof Let $f : [a,b] \times [0,\infty)$ be given by $f(x,y) = e^{-xt}$. Then f is continuous (hence measurable) and $f > 0$. By Toneli's theorem

$$\int_0^\infty \int_a^b e^{-xt} dx\, dt = \int_a^b \int_0^\infty e^{-xt} dt\, dx.$$

But,

$$\int_0^\infty \int_a^b e^{-xt} dx\, dt = \int_0^\infty (e^{-at} - e^{-bt}) \frac{1}{t} dt,$$

and

$$\int_a^b \int_0^\infty e^{-xt} dt\, dx = \log(b/a).$$

The result thus follows.

4. Let $(E, \mathcal{B}, \mu)$ be a measure space. Show that μ is σ-finite **if and only if** there exists a **strictly** positive measurable function $f \in L^1(\mu)$.

Proof Suppose μ is σ-finite. Then $E = \bigcup_{n=1}^\infty E_n$, where $\{E_n\}$ is a family of measurable pairwise disjoint sets such that $\mu(E_n) < \infty$ for all n. Define $f : E \to \mathbb{R}$ by $f(x) = \sum_{n=1}^\infty \frac{2^{-n}}{\mu(E_n)} 1_{E_n}$. Then f is a strictly positive measurable function. Furthermore,

$$\int_E f\, d\mu = \sum_{n=1}^\infty \int_{E_n} f\, d\mu = \sum_{n=1}^\infty \int_{E_n} \frac{2^{-n}}{\mu(E_n)}\, d\mu = \sum_{n=1}^\infty 2^{-n} < \infty.$$

Thus $f \in L^1(\mu)$.

Conversely, suppose there exists a **strictly** positive measurable function $f \in L^1(\mu)$. Let $F_n = \{f \geq \frac{1}{n}\}$. Then, $\{F_n\}$ is an increasing sequence of measurable sets such that $E = \bigcup_{n=1}^\infty F_n$, and by Markov inequality $\mu(F_n) \leq n \int f\, d\mu < \infty$. Set $E_1 = F_1$ and $E_n = F_n \setminus F_{n-1}$, then E_n are pairwise disjoint, $E = \bigcup_{n=1}^\infty E_n$ and $\mu(E_n) \leq \mu(F_n) < \infty$. Thus, μ is σ-finite.