Universiteit Utrecht
Mathematisch Instituut

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Measure and Integration Final Exam

Due date: June 30, 2004

You must work on this exam individually. It is not allowed to discuss this exam with your fellow student.

1. Let ν be a σ-finite measure on $(E, \mathcal{B})$, and suppose $E = \bigcup_{n=1}^{\infty} E_n$, where $\{E_n\}$ is a collection of pairwise disjoint measurable sets such that $\nu(E_n) < \infty$ for all $n \geq 1$. Define μ on $\mathcal{B}$ by $\mu(\Gamma) = \sum_{n=1}^{\infty} 2^{-n}\nu(\Gamma \cap E_n)/(\nu(E_n) + 1)$.
   (a) Prove that μ is a finite measure on $(E, \mathcal{B})$.
   (b) Show that for any $\Gamma \in \mathcal{B}$, $\nu(\Gamma) = 0$ **if and only if** $\mu(\Gamma) = 0$.
   (c) Find explicitly two positive measurable functions f and g such that
   $$\mu(A) = \int_A f d\nu \text{ and } \nu(A) = \int_A g d\mu$$
   for all $A \in \mathcal{B}$.

2. Suppose that μ_i, ν_i are finite measures on $(E, \mathcal{B})$ with $\mu_i \ll \nu_i$, $i = 1, 2$. Let $\nu = \nu_1 \times \nu_2$ and $\mu = \mu_1 \times \mu_2$.
   (a) Show that $\mu \ll \nu$.
   (b) Prove that $\frac{d\mu}{d\nu}(x, y) = \frac{d\mu_1}{d\nu_1}(x) \cdot \frac{d\mu_2}{d\nu_2}(y)$ ν a.e.

3. Let $(E, \mathcal{B}, \mu)$ be a measure space.
   (a) Suppose $f, g \in L^1(\mu)$ are such that $\int_A f d\mu \leq \int_A g d\mu$ for all $A \in \mathcal{B}$. Show that $f \leq g$ μ a.e.
   (b) Show that μ is σ-finite **if and only if** there exists a **strictly** positive measurable function $f \in L^1(\mu)$.

4. Let $(E, \mathcal{B}, \mu)$ be a measure space, and $\{f_n\} \subseteq L^1(\mu)$, $f \in L^1(\mu)$ be such that (i) $f_n, f \geq 0$ for $n \geq 1$, (ii) $\int_E f_n d\mu = \int_E f d\mu < \infty$ for $n \geq 1$, and (iii) $f_n \to f$ μ a.e.
   (a) Show that $\lim_{n\to\infty} \int_E (f - f_n)^+ d\mu = 0$.
   (b) Prove that $\lim_{n\to\infty} \sup_{A\in\mathcal{B}} |\int_A f_n \, d\mu - \int_A f \, d\mu| = 0$.

5. Consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$, where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra, and λ Lebesgue measure.

   (a) Let μ be a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R})$ and $f : \mathbb{R} \to [0, 1)$ a measurable function such that $\mu\left(f^{-1}([\frac{k}{2^n}, \frac{k+1}{2^n}))\right) = \frac{1}{2^n}$ for $n \geq 1$ and $k = 0, \cdots, 2^n - 1$. Show that $f \in L^2(\mu)$, and determine the value of $||f||_{L^2(\mu)}$.

   (b) Show that $\lim_{n\to\infty} \int_0^n \left(1 - \frac{x}{n}\right)^n e^{x/2} d\lambda(x) = 2$.

   (c) Let $f : \mathbb{R} \to \overline{\mathbb{R}}$ be measurable, and suppose $\int_{\mathbb{R}} f(x) d\lambda(x)$ exists. Show that for all $a \in \mathbb{R}$, one has

$$\int_{\mathbb{R}} f(x - a) d\lambda(x) = \int_{\mathbb{R}} f(x) d\lambda(x).$$

   (d) Let $k, g \in L^1(\lambda)$. Define $F : \mathbb{R}^2 \to \mathbb{R}$, and $h : \mathbb{R} \to \overline{\mathbb{R}}$ by

$$F(x, y) = k(x - y)g(y) \text{ and } h(x) = \int_{\mathbb{R}} F(x, y) d\lambda(y).$$

      (i) Show that F is measurable.
      (ii) Show that $\lambda(|h| = \infty) = 0$ and $\int_{\mathbb{R}} |h| d\lambda \leq \left(\int_{\mathbb{R}} |k| d\lambda\right) \left(\int_{\mathbb{R}} |g| d\lambda\right)$.

6. Consider the measure space $([a, b], \mathcal{B}, \lambda)$, where $\mathcal{B}$ is the Borel σ-algebra on $[a, b]$, andλ is the restriction of the Lebesgue measure on $[a, b]$. Let $f : [a, b] \to \mathbb{R}$ be a bounded Riemann integrable function. Show that the Riemann integral of f on $[a, b]$ is equal to the Lebesgue integral of f on $[a, b]$, i.e.

$$(R) \int_a^b f(x) dx = \int_{[a,b]} f d\lambda.$$