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Measure and Integration 2006-Selected Solutions 12

1. (**Exercise 12.1, p.116**) Let $(X, \mathcal{A}, \mu)$ be a finite measure space, and let $1 \leq q < p < \infty$.

   (i) Show that if $u \in \mathcal{L}^p(\mu)$, then $||u||_q \leq \mu(X)^{\frac{1}{q}-\frac{1}{p}}||u||_p$.

   (ii) Conclude that $\mathcal{L}^p(\mu) \subset \mathcal{L}^q(\mu)$ for all $p \geq q \geq 1$, and that an $\mathcal{L}^p(\mu)$-Cauchy sequence is also $\mathcal{L}^q(\mu)$-Cauchy.

   (iii) Is part (ii) true if μ is **not** finite?

   Proof (i): Note that if $u \in \mathcal{L}^p(\mu)$, then $u^q \in \mathcal{L}^{\frac{p}{q}}(\mu)$, and $\frac{p}{q} > 1$. Further, if $r = \frac{p}{q}$, then the conjugate of r is $s = \frac{p}{p-q}$ (i.e., $\frac{1}{r} + \frac{1}{s} = 1$). Applying Hölders's inequality to the functions $u^q \in \mathcal{L}^{\frac{p}{q}}(\mu)$, and $1 \in \mathcal{L}^{\frac{p}{p-q}}(\mu)$ (since μ is a finite measure), we get

$$\begin{aligned} ||u||_q^q = \int |u|^q\,d\mu &\leq \left(\int (|u|^q)^{\frac{p}{q}}\,d\mu\right)^{\frac{q}{p}} \left(\int 1^{\frac{p}{p-q}}\,d\mu\right)^{\frac{p-q}{p}} \\ &= \left(\int (|u|^p)\,d\mu\right)^{\frac{q}{p}} (\mu(X))^{1-\frac{q}{p}} \\ &= ||u||_p^q\,(\mu(X))^{1-\frac{q}{p}}. \end{aligned}$$

   Hence, $||u||_q \leq \mu(X)^{\frac{1}{q}-\frac{1}{p}}||u||_p$.

   Proof (ii): Suppose $u \in \mathcal{L}^p(\mu)$, then $||u||_p < \infty$. Since $\mu(X) < \infty$, then by part (i) we have that $||u||_q < \infty$ so that $u \in \mathcal{L}^q(\mu)$. This shows that $\mathcal{L}^p(\mu) \subset \mathcal{L}^q(\mu)$. Finally suppose $(u_n) \subset \mathcal{L}^p(\mu)$ is $\mathcal{L}^p(\mu)$-Cauchy, by part (i),

$$||u_n - u_m||_q \leq ||u_n - u_m||_p \mu(X)^{\frac{1}{q}-\frac{1}{p}} \to 0 \text{ as } m, n \to \infty.$$

   Hence, (u_n) is $\mathcal{L}^q(\mu)$-Cauchy.

   Proof (iii): The result is not true if μ is not a finite measure. Consider for example $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$, where λ is Lebesgue measure. Let $f = \dfrac{1}{x} \cdot 1_{(1,\infty)}$. Then $\int_{\mathbb{R}} f\,d\lambda = \infty$, while $\int_{\mathbb{R}} f^2\,d\lambda = 1$. This shows that $f \in \mathcal{L}^2(\lambda)$ but $f \notin \mathcal{L}^1(\lambda)$. In general for any $q < p$, choose $\frac{1}{p} < \alpha < \frac{1}{q}$ and consider the function $g(x) = \frac{1}{x^\alpha}$, then $g \in \mathcal{L}^p(\lambda)$, but $g \notin \mathcal{L}^q(\lambda)$.

2. (**Exercise 12.6, p.116**) Let $1 \leq p < \infty$ and $u, u_k \in \mathcal{L}^p(\mu)$ such that $\sum_{k=1}^{\infty} ||u - u_k||_p < \infty$. Show that $\lim_{k\to\infty} u_k(x) = u(x)\ \mu$ a.e.

Proof: Since $\sum_{k=1}^{\infty} ||u - u_k||_p < \infty$, it follows that $\lim_{k\to\infty} ||u - u_k||_p = 0$, that is $\mathcal{L}^p(\mu) - \lim_{k\to\infty} u_k = u$. By Corollary 2.8, there exists a subsequence $(u_{n(k)}) \subset (u_k)$ which converges μ a.e. to u, i.e. $\lim_{k\to\infty} u_{n(k)}(x) = u(x)\, \mu$ a.e.

We now show that the series $\sum_{j=0}^{\infty}(u_{j+1}(x) - u_j(x))$ is finite μ a.e. ($u_0 = 0$) by showing that it is absolutely convergent μ a.e. From Lemma 12.6 and Minkowski's inequality, we have

$$\begin{aligned} || \textstyle\sum_{j=0}^{\infty} |u_{j+1} - u_j| ||_p &\leq \textstyle\sum_{j=1}^{\infty} ||u_{j+1} - u_j||_p \\ &\leq \textstyle\sum_{j=0}^{\infty} ||u_{j+1} - u||_p + \sum_{j=0}^{\infty} ||u_j - u||_p < \infty. \end{aligned}$$

By by Corollary 10.13, we have $\sum_{j=0}^{\infty} |u_{j+1}(x) - u_j(x)| < \infty$ μ a.e. and hence $\sum_{j=0}^{\infty}(u_{j+1}(x) - u_j(x)) < \infty$ μ a.e. Furthermore,

$$\lim_{j\to\infty} u_j(x) = \lim_{j\to\infty} \sum_{k=0}^{j-1}(u_{k+1}(x) - u_k(x)) = \sum_{k=0}^{\infty}(u_{k+1}(x) - u_k(x))\ \mu \text{ a.e.}$$

Finally, $\sum_{k=0}^{\infty}(u_{k+1}(x) - u_k(x)) = \lim_{j\to\infty} u_j(x) = \lim_{k\to\infty} u_{n(k)}(x) = u(x)$ μ a.e.

3. (**Exercise 12.7, p.116**) Consider $([0,1], \mathcal{B}, \lambda)$, where λ is Lebesgue measure restricted to $[0,1]$. Show that the sequence $u_n(x) = n \cdot \mathbf{1}_{(0,\frac{1}{n})}$, $n \in \mathbb{N}$ converges pointwise to $u(x) = 0$, but no subsequence of (u_n) converges in $\mathcal{L}^p(\lambda)$ for any $p \geq 1$.

   Proof: If $x = 0$ or 1, then $u_n(0) = 0 = u_n(1)$ for all n hence $\lim_{n\to\infty} u_n(0) = 0 = \lim_{n\to\infty} u_n(1)$. Suppose $0 < x < 1$, then there exists an integer $N > 1$ such that $\frac{1}{N} < x$. Then for any $n \geq N$, we have $u_n(x) = 0$. Thus, $\lim_{n\to\infty} u_n(x) = 0$. Therefore, the sequence u_n converges pointwise to 0 for all $x \in [0,1]$.

   For any subsequence $(u_{n(j)})$ of (u_n), we have

$$||u_{n(j)}||_p^p = \int_{[0,1]} |u_{n(j)}|^p \, d\lambda = n(j)^{p-1} \longrightarrow_{j\to\infty} \begin{cases} 1 & p = 1 \\ \infty & p > 1 \end{cases}$$

   Hence, $\lim_{j\to\infty} ||u_{n(j)}||_p \neq 0$, i.e. $\mathcal{L}^p(\lambda) - \lim_{j\to\infty} u_{n(j)} \neq 0$. In fact no subsequence has a limit point in $\mathcal{L}^p(\lambda)$. For suppose $w = \mathcal{L}^p(\lambda) - \lim_{j\to\infty} u_{n(j)}$, then by Corollary 12.8 there exists a subsequence $(u_{n'_{n(j)}})$ of $(u_{n(j)})$ which converges μ a.e. to w. But since (u_j) converges to 0 μ a.e. (in fact for every point in $[0,1]$), it follows that $w = 0$ which is a contradiction.

4. (**Exercise 12.10, p.116**) Let $(X, \mathcal{A}, \mu)$ be a finite measure space. Show that every measurable function $u \geq 0$ with $\int exp(hu(x))\, d\mu(x) < \infty$ for some $h \geq 0$ is in $\mathcal{L}^p(\mu)$ for all $p \geq 1$.

   Proof: Notice that $exp(hu(x)) = \sum_{n=0}^{\infty} \frac{h^n u^n(x)}{n!}$. Since $u(x) \geq 0$ and $h \geq 0$ we have $\frac{h^n u^n(x)}{n!} < exp(hu(x))$ for all $n \in \mathbb{N}$. Thus, $\int u^n \, d\mu < \infty$ and $u \in \mathcal{L}^n(\mu)$ for all $n \in \mathbb{N}$.

Finally, for any $p \geq 1$ a non-integer, there exist an integer n such that $p < n$. Then, by exercise 12.1(ii) we have $\mathcal{L}^n(\mu) \subset \mathcal{L}^p(\mu)$. Hence, $u \in \mathcal{L}^p(\mu)$ for all $p \geq 1$.