

## Measure and Integration 2012-Selected Solutions Chapter 3

- 1. (Exercise 3.11, p. 21) According to the book, a family  $\mathcal{M} \subset \mathcal{P}(X)$  is a monotone class if it is closed under countable unions and countable intersections.
  - (i) Show that for every family  $\mathcal{E} \subset \mathcal{P}(X)$ , there is a smallest monotone class  $m(\mathcal{E})$  containing  $\mathcal{E}$ .
  - (ii) Assume  $\emptyset \in \mathcal{E}$ , and  $E \in \mathcal{E} \Rightarrow E^c \in \mathcal{E}$ . Show that  $\Sigma = \{B \in m(\mathcal{E}) : B^c \in m(\mathcal{E})\}$  is a  $\sigma$ -algebra.
  - (iii) Assume that the assumptions in part (ii) hold, show that  $m(\mathcal{E}) = \sigma(\mathcal{E})$ .

**Proof (i)** Define  $m(\mathcal{E}) = \bigcap \{\mathcal{D} : \mathcal{D} \text{ is a monotone class and } \mathcal{E} \subset \mathcal{D} \}$ . Since  $\mathcal{P}(X)$  is a monotone class containing  $\mathcal{E}, m(\mathcal{E})$  is well-defined. It remains to show that  $m(\mathcal{E})$ is a monotone class. Let  $(A_n) \subset m(\mathcal{E})$ , then  $(A_n) \subset \mathcal{D}$  for each monotone class containing  $\mathcal{E}$ . Thus,  $\bigcup_n A_n, \bigcap_n A_n \in \mathcal{D}$  for all  $\mathcal{D}$ . Therefore,  $\bigcup_n A_n, \bigcap_n A_n \in m(\mathcal{E})$ .

**Proof (ii)** Since  $\emptyset \in \mathcal{E}$ , and  $\mathcal{E}$  is closed under complements, we have that  $\emptyset, X \in \mathcal{E} \subset m(\mathcal{E})$ . This implies that  $X \in \Sigma$ . From the definition of  $\Sigma$ , it is clear that if  $B \in \Sigma$ , then  $B^c \in \Sigma$ . Finally, let  $(B_n) \in \Sigma$ . Then,  $B_n, B_n^c \in \Sigma$  for all n. Since  $m(\mathcal{E})$  is closed under countable unions and intersections, it follows that  $\bigcup_n B_n \in m(\mathcal{E})$ , and  $(\bigcup_n B_n)^c = \bigcap_n B_n^c \in m(\mathcal{E})$ . Thus,  $\bigcup_n B_n \in \Sigma$ , and  $\Sigma$  is a  $\sigma$ -algebra.

**Proof (iii)** From part (ii) and the fact that  $\sigma(\mathcal{E})$  is also a monotone class containing  $\mathcal{E}$ , we see that  $\mathcal{E} \subset \Sigma \subset m(\mathcal{E}) \subset \sigma(\mathcal{E})$ . Since  $\sigma(\mathcal{E})$  is the smallest  $\sigma$ -algebra containing  $\mathcal{E}$ , we have  $\sigma(\mathcal{E}) \subset \Sigma \subset m(\mathcal{E})$ . Therefore,  $\sigma(\mathcal{E}) = m(\mathcal{E})$ .

- 2. (Exercise 3.12, p. 21). The aim of this problem is to show that  $\mathcal{B}(\mathbb{R}^n) = \mathcal{M}$ , where  $\mathcal{B}(\mathbb{R}^n)$  is the Borel  $\sigma$ -algebra over  $\mathbb{R}^n$ , and  $\mathcal{M}$  is the smallest monotone class containing the open sets  $\mathcal{O}^n$ . Show that
  - (i)  $\mathcal{M}$  is well-defined, and  $\mathcal{M} \subset \sigma(\mathcal{O}^n)$ .
  - (ii) If  $U \in \mathcal{M}$ , then  $U^c \in \mathcal{M}$ , i.e.  $\mathcal{M}$  contains all closed sets.
  - (iii)  $\{B \in \mathcal{M} : B^c \in \mathcal{M}\}$  is a  $\sigma$ -algebra.
  - (iv)  $\sigma(\mathcal{O}^n) \subset \{B \in \mathcal{M} : B^c \in \mathcal{M}\} \subset \mathcal{M}.$

**Proof (i)** By problem 3.11,  $\mathcal{M} = \bigcap \{ \mathcal{D} : \mathcal{D} \text{ is a monotone class and } \mathcal{O}^n \subset \mathcal{D} \}$ . Since  $\sigma(\mathcal{O}^n) = \mathcal{B}(\mathbb{R}^n)$  is also a monotone class containing  $\mathcal{O}^n$ , we have that  $\mathcal{M} \subset \sigma(\mathcal{O}^n)$ . **Proof (ii)** Let F be a closed set. We show that  $F \in \mathcal{M}$  by showing that F is the countable intersection of open sets. To this end, set  $U_n = \{y \in \mathbb{R}^n : |x - y| < \frac{1}{n} \text{ for some } x \in F\}$ . We first show that  $U_n$  is open by showing that  $U_n^c$  is closed. Notice that  $U_n^c = \{y \in \mathbb{R}^n : |x - y| \ge \frac{1}{n} \text{ for all } x \in F\}$ . Now, let  $(y_n)$  be a sequence in  $U_n^c$  converging to y. We must show that  $y \in U_n^c$ . Let  $\epsilon > 0$ , then there exists an integer M > 0 such that  $|y_m - y| < \epsilon$  for all  $m \ge M$ . Pick any  $m \ge M$ , then for all  $x \in F$ 

$$|x-y| \ge |y_m - x| - |y - y_m| \ge \frac{1}{n} - \epsilon.$$

Since  $\epsilon > 0$  is arbitrary, it follows that  $|x - y| \ge \frac{1}{n}$  for all  $x \in F$ . Hence,  $y \in U_n^c$ . This shows that  $U_n^c$  is closed, and hence  $U_n$  is open. We claim that  $F = \bigcap_{n \in \mathbb{N}} U_n$ . Clearly,  $F \subseteq U_n$  for all n, hence  $F \subset \bigcap_{n \in \mathbb{N}} U_n$ . Now suppose  $y \in \bigcap_{n \in \mathbb{N}} U_n$ , then for each m there exists  $x_m \in F$  such that  $|x_m - y| < 1/m$ . Then,  $(x_m)$  is a sequence in F converging to y. Since F is closed, this implies that  $y \in F$ . Thus  $\bigcap_{n \in \mathbb{N}} U_n \subset F$ . Therefore,  $E = \bigcap_{n \in \mathbb{N}} U_n$  and  $E \in \mathcal{M}$ .

Therefore,  $F = \bigcap_{n \in \mathbb{N}} U_n$ , and  $F \in \mathcal{M}$ .

**Proof (iii)** Let  $\mathcal{A} = \{B \in \mathcal{M} : B^c \in \mathcal{M}\}$ . Since  $\emptyset, \mathbb{R}^n \in \mathcal{O}^n \subset \mathcal{M}$ , it follows that  $\mathbb{R}^n \in \mathcal{A}$ . From the definition of  $\mathcal{A}$ , it is clear that if  $B \in \mathcal{A}$ , then  $B^c \in \mathcal{A}$ . Finally, let  $(B_n) \in \mathcal{A}$ . Then,  $B_n, B_n^c \in \mathcal{M}$  for all n. Since  $\mathcal{M}$  is closed under countable unions and intersections, it follows that  $\cup_n B_n \in \mathcal{M}$ , and  $(\cup_n B_n)^n = \cap_n B_n^c \in \mathcal{M}$ . Thus,  $\cup_n B_n \in \mathcal{A}$ , and  $\mathcal{A}$  is a  $\sigma$ -algebra.

**Proof (iv)** From the definition of  $\mathcal{M}$  and parts (ii) and (iii), we see that  $\mathcal{O}^n \subset \mathcal{A}$ . Hence  $\mathcal{B}(\mathbb{R}^n) = \sigma(\mathcal{O}^n) \subset \mathcal{A} \subset \mathcal{M}$ . Together with part (i), we see that  $\mathcal{B}(\mathbb{R}^n) = \mathcal{M}$ .