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Measure and Integration Solutions Extra Exercises Final 2008

(1) Consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$, where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra, and λ Lebesgue measure.

(a) Let $f \in \mathcal{L}^1(\lambda)$. Show that for all $a \in \mathbb{R}$, one has

$$\int_{\mathbb{R}} f(x-a) d\lambda(x) = \int_{\mathbb{R}} f(x) d\lambda(x).$$

(b) Let $k, g \in \mathcal{L}^1(\lambda)$. Define $F : \mathbb{R}^2 \to \mathbb{R}$, and $h : \mathbb{R} \to \overline{\mathbb{R}}$ by

$$F(x,y) = k(x-y)g(y) \text{ and } h(x) = \int_{\mathbb{R}} F(x,y) d\lambda(y).$$

(i) Show that F is measurable.

(ii) Show that

$$\int_{\mathbb{R}} |h(x)| d\lambda(x) \leq \left(\int_{\mathbb{R}} |k(x)| d\lambda(x) \right) \left(\int_{\mathbb{R}} |g(y)| d\lambda(y) \right).$$

and $\lambda(|h| = \infty) = 0$.

Proof(a): We apply the standard argument. Suppose first that $f = \mathbf{1}_A$, where $A \in \mathcal{B}(\mathbb{R})$. By translation invariance of Lebesgue measure, we have for any $a \in \mathbb{R}$

$$\int \mathbf{1}_A(x)\, d\lambda(x) = \lambda(A) = \lambda(A) = \lambda(A+a) = \int \mathbf{1}_{A+a}(x)\, d\lambda(x) = \int \mathbf{1}_A(x-a)\, d\lambda(x).$$

Hence the result is true for indicator functions (we do not even need that $\lambda(A) < \infty$). Suppose now that $f \in \mathcal{E}^+$, and let $f = \sum_{i=0}^n a_i \mathbf{1}_{A_i}$ be a standard representation. Then

$$\int f(x)\, d\lambda(x) = \sum_{i=0}^n a_i \int \mathbf{1}_{A_i}(x)\, d\lambda(x) = \sum_{i=0}^n a_i \int \mathbf{1}_{A_i}(x-a)\, d\lambda(x) = \int f(x-a)\, d\lambda(x).$$

Now let f be any non-negative measurable function. Then, there exists an increasing sequence $(g_n) \in \mathcal{E}^+$ converging (pointwise) to f. By Beppo-Levi, we have

$$\int f(x)\, d\lambda(x) = \lim_{n\to\infty} \int g_n(x)\, d\lambda(x) = \lim_{n\to\infty} \int g_n(x-a)\, d\lambda(x) = \int f(x-a)\, d\lambda(x).$$

Finally, suppose $f \in \mathcal{L}^1(\lambda)$. Write $f = f^+ - f^-$. Since $f^+, f^- \geq 0$, then

$$\begin{aligned} \int f(x)\, d\lambda(x) &= \int f^+(x)\, d\lambda(x) - \int f^-(x)\, d\lambda(x) \\ &= \int f^+(x-a)\, d\lambda(x) - \int f^-(x-a)\, d\lambda(x) = \int f(x-a)\, d\lambda(x). \end{aligned}$$

(Note that only in the last part is the integrability of f needed).

Proof(b)(i): To show measurablity of F, we first extend the domain of g to $\mathbb{R}^2$ as follows. Define $\overline{g} : \mathbb{R}^2 \to \mathbb{R}$ by $\overline{g}(x,y) = g(y)$. It is easy to see that $\overline{g}$ is $\mathcal{B}(\mathbb{R}^2)/\mathcal{B}(\mathbb{R})$ measurable. Moreover, the function $d : \mathbb{R}^2 \to \mathbb{R}$ given by $d(x,y) = x - y$ is continuous hence $\mathcal{B}(\mathbb{R}^2)/\mathcal{B}(\mathbb{R})$ measurable. Since

$$F(x,y) = k(x-y)g(y) = k \circ d(x,y)\overline{g}(x,y)$$

is the product of two $\mathcal{B}(\mathbb{R}^2)/\mathcal{B}(\mathbb{R})$ measurable functions, it follows that F is $\mathcal{B}(\mathbb{R}^2)/\mathcal{B}(\mathbb{R})$ measurable.

Proof(b)(ii): By part (a), we have

$$\begin{aligned}\int\int |F(x,y)|\, d\lambda(x) d\lambda)(y) &= \int\int |k(x-y)||g(y)|\, d\lambda(x) d\lambda)(y)\\ &= \int\int |k(x)||g(y)|\, d\lambda(x) d\lambda)(y)\\ &= \int |k(x)|\, d\lambda(x) \int |g(y)| d\lambda)(y) < \infty.\end{aligned}$$

By Fubini's Theorem, this implies that F is $\lambda \times \lambda$ integrable, and

$$\begin{aligned}\int |h(x)|\, d\lambda(x) &= \int |\int F(x,y)\, d\lambda(y)|\, d\lambda(x)\\ &\leq \int\int |F(x,y)|\, d\lambda(y)\, d\lambda(x)\\ &= \int\int |F(x,y)|\, d\lambda(x) d\lambda)(y)\\ &= \int |k(x)|\, d\lambda(x) \int |g(y)| d\lambda)(y) < \infty.\end{aligned}$$

Since $\int |h(x)|\, d\lambda(x) < \infty$, it follows that $\lambda(|h| = \infty) = 0$.

(2) Consider the measure space $((0,\infty), \mathcal{B}((0,\infty), \lambda)$, where $\mathcal{B}((0,\infty))$ and λ are the restrictions of the Borel σ-algebra and Lebesgue measure to the interval $(0,\infty)$. Show that

$$\lim_{n\to\infty} \int_{(0,n)} \left(1 + \frac{x}{n}\right)^n e^{-2x}\, d\lambda(x) = 1.$$

Proof: Let $u_n(x) = \mathbf{1}_{(0,n)} \left(1 + \frac{x}{n}\right)^n e^{-2x}$, then $\lim_{n\to\infty} u_n(x) = \mathbf{1}_{(0,\infty)} e^{-x}$. Using the fact that $1 + x \leq e^x$, we see that $u_n(x) \leq \mathbf{1}_{(0,\infty)} e^{-x}$. Since the function e^{-x} is Riemann integrable on $[0,\infty)$, it follows that it is Lebesgue integrable on $[0,\infty)$ (and hence also on $(0,\infty)$). By Lebesgue Dominated Convergence Theorem (or the Monotone Convergence Theorem), we have

$$\begin{aligned}\lim_{n\to\infty} \int_{(0,n)} \left(1 + \frac{x}{n}\right)^n e^{-2x}\, d\lambda(x) &= \lim_{n\to\infty} \int u_n(x) d\lambda(x)\\ &= \int \mathbf{1}_{(0,\infty)} e^{-x} d\lambda(x) = \int_0^\infty e^{-x}\, dx = 1.\end{aligned}$$

(3) Let $(X, \mathcal{A}, \mu)$ be a probability space (i.e. $\mu(X) = 1$).
  (a) Suppose $1 \leq p < r$, and $f_n, f \in \mathcal{L}^r(\mu)$ satisfy $\lim\limits_{n\to\infty} ||f_n - f||_r = 0$. Show that $\lim\limits_{n\to\infty} ||f_n - f||_p = 0$.
  (b) Assume $p, q > 1$ satisfy $1/p + 1/q = 1$. Suppose $f_n, f \in \mathcal{L}^p(\mu)$, and $g_n, g \in \mathcal{L}^q(\mu)$ satisfy

$$\lim_{n\to\infty} ||f_n - f||_p = \lim_{n\to\infty} ||g_n - g||_q = 0.$$

  Show that $\lim\limits_{n\to\infty} ||f_n g_n - fg||_1 = 0$.

Proof(a): Since $\mu(X) = 1$, by problem 12.1, we have

$$0 \leq \lim_{n\to\infty} ||f_n - f||_p \leq \lim_{n\to\infty} ||f_n - f||_r = 0.$$

Thus, $\lim\limits_{n\to\infty} ||f_n - f||_p = 0$.

Proof(b): First notice that by the triangle inequality for $||\ ||_p$, we have

$$\lim_{n\to\infty} |\, ||f_n||_p - ||f||_p\, | \leq \lim_{n\to\infty} ||f_n - f||_p = 0.$$

Thus, $\lim_{n\to\infty} ||f_n||_p = ||f||_p$. By Holder's inequality we have,

$$\begin{aligned}||f_n g_n - fg||_1 &= \int |f_n g_n - fg|\, d\mu\\ &\leq \int |f_n|\, |g_n - g|\, d\mu + \int |g|\, |f_n - f|\, d\mu\\ &\leq ||f_n||_p ||g_n - g||_q + ||g||_q ||f_n - f||_p.\end{aligned}$$

Taking limits, we get the desired result.

(4) Let $0 < a < b$. Prove with the help of Tonelli's theorem (applied to the function $f(x,y) = e^{-xt}$) that $\int_{[0,\infty)} (e^{-at} - e^{-bt})\frac{1}{t} d\lambda(t) = \log(b/a)$, where λ denotes Lebesgue measure.

Proof Let $f : [a,b] \times [0,\infty)$ be given by $f(x,y) = e^{-xt}$. Then f is continuous (hence measurable) and $f > 0$. By Toneli's theorem

$$\int_{[0,\infty)} \int_{[a,b]} e^{-xt} d\lambda(x)\, d\lambda(t) = \int_{[a,b]} \int_{[0,\infty)} e^{-xt} d\lambda(t)\, d\lambda(x).$$

For each fixed $x \in [a,b]$, the function $t \to e^{-xt}$ is Riemann integrable on $[0,\infty)$, so that

$$\int_{[0,\infty)} e^{-xt} d\lambda(t) = \int_0^\infty e^{-xt} dt = \frac{1}{x}.$$

Furthermore, the function $x \to \frac{1}{x}$ is Riemann integrable on $[a,b]$, thus

$$\int_{[a,b]} \int_{[0,\infty)} e^{-xt} d\lambda(t)\, d\lambda(x) = \int_{[a,b]} \frac{1}{x}\, d\lambda(x) = \int_a^b \frac{1}{x}\, dx = \log(b/a).$$

On the other hand,

$$\int_{[0,\infty)} \int_{[a,b]} e^{-xt} d\lambda(x)\, d\lambda(t) = \int_{[0,\infty)} \int_a^b e^{-xt} dx\, d\lambda(t) = \int_{[0,\infty)} (e^{-at} - e^{-bt})\frac{1}{t} d\lambda(t).$$

Therefore, $\int_{[0,\infty)} (e^{-at} - e^{-bt})\frac{1}{t} d\lambda(t) = \log(b/a)$.

(5) Let $(E, \mathcal{B}, \nu)$ be a measure space, and $h : E \to \mathbb{R}$ a non-negative measurable function. Define a measure μ on $(E, \mathcal{B})$ by $\mu(A) = \int_A h d\nu$ for $A \in \mathcal{B}$. Show that for every non-negative measurable function $F : E \to \mathbb{R}$ one has

$$\int_E F\, d\mu = \int_E F h\, d\nu.$$

Conclude that the result is still true for $F \in \mathcal{L}^1(\mu)$ which is not necessarily non-negative.

Proof Suppose first that $F = 1_A$ is the indicator function of some measurable set $A \in \mathcal{B}$. Then,

$$\int_E F\, d\mu = \mu(A) = \int_A h\, d\nu = \int_E 1_A h d\nu = \int_E F h d\nu.$$

Suppose now that $F = \sum_{k=1}^n \alpha_k 1_{A_k}$ is a non-negative measurable step function. Then,

$$\int_E F\, d\mu = \sum_{k=1}^n \alpha_k \mu(A_k) = \sum_{k=1}^n \alpha_k \int_E 1_A h d\nu = \int_E \sum_{k=1}^n \alpha_k 1_A h d\nu = \int_E F h d\nu.$$

Suppose that F is a non-negative measurable function, then there exists a sequence of non-negative measurable step functions F_n such that $F_n \uparrow F$. Then, $F_n h \uparrow Fh$, and by Beppo-Levi,

$$\int_E F\, d\mu = \lim_{n\to\infty} \int_E F_n\, d\mu = \lim_{n\to\infty} \int_E F_n h d\nu = \int_E F h d\nu.$$

Finally, suppose that $F \in \mathcal{L}^1(\mu)$. Since F^+, F^- are non-negative, we have

$$\int_E F^+\, d\mu = \int_E F^+ h\, d\nu \text{ and } \int_E F^-\, d\mu = \int_E F^- h\, d\nu.$$

Since $F \in \mathcal{L}^1(\mu)$, from the above we see that $Fh \in \mathcal{L}^1(\nu)$, hence

$$\int_E F\, d\mu = \int_E F^+\, d\mu - \int_E F^-\, d\mu = \int_E F^+ h\, d\nu - \int_E F^- h\, d\nu = \int_E F h\, d\nu.$$

(6) Let $(X, \mathcal{A}, \mu_1)$ and $(Y, \mathcal{B}, \nu_1)$ be σ-finite measure spaces. Suppose $f \in \mathcal{L}^1(\mu_1)$ and $g \in \mathcal{L}^1(\nu_1)$ are non-negative. Define measures μ_2 on $\mathcal{A}$ and ν_2 on $\mathcal{B}$ by

$$\mu_2(A) = \int_A f\, d\mu_1 \text{ and } \nu_2(B) = \int_B g\, d\nu_1,$$

for $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

(a) For $D \in \mathcal{A} \otimes \mathcal{B}$ and $y \in Y$, let $D_y = \{x \in X : (x, y) \in D\}$. Show that if $\mu_1(D_y) = 0$ ν_1 a.e., then $\mu_2(D_y) = 0$ ν_2 a.e.
(b) Show that if $D \in \mathcal{A} \otimes \mathcal{B}$ is such that $(\mu_1 \times \nu_1)(D) = 0$ then $(\mu_2 \times \nu_2)(D) = 0$.
(c) Show that for every $D \in \mathcal{A} \otimes \mathcal{B}$ one has

$$(\mu_2 \times \nu_2)(D) = \int_D f(x)g(y)\, d(\mu_1 \times \nu_1)(x, y).$$

Proof(a) Suppose $\mu_1(D_y) = 0$ ν_1 a.e. Let $B = \{y \in Y : \mu_1(D_y) > 0\}$, and $C = \{y \in Y : \mu_2(D_y) > 0\}$. By our assumption, $\nu_1(B) = 0$. By Theorem 10.9(ii), for any $y \in Y \setminus B$ one has $\mu_2(D_y) = 0$. Thus, $C \subset B$, so that $\nu_1(C) = 0$. Applying Theorem 10.9(ii) again, we see that $\nu_2(C) = 0$. Thus, $\mu_2(D_y) = 0$ ν_2 a.e.

Proof(b) Suppose that $D \in \mathcal{A} \otimes \mathcal{B}$ is such that $(\mu_1 \times \nu_1)(D) = 0$. Then,

$$\int \mu_1(D_y)\, d\nu_1(y) = (\mu_1 \times \nu_1)(D) = 0.$$

By Theorem 10.9(i), we have that $\mu_1(D_y) = 0$ ν_1 a.e. By part (a) above this implies that $\mu_2(D_y) = 0$ ν_2 a.e. Thus, by Theorem 10.9(i)

$$(\mu_2 \times \nu_2)(D) = \int \mu_2(D_y)\, d\nu_2(y) = 0.$$

Proof(c) By Tonelli's Theorem, and problem 5, we have

$$\begin{aligned}
(\mu_2 \times \nu_2)(D) &= \int_Y \int_X \mathbf{1}_{D_y}(x)\, d\mu_2(x)\, d\nu_2(y) \\
&= \int_Y \left(\int_X \mathbf{1}_{D_y}(x) f(x)\, d\mu_1(x) \right) d\nu_2(y) \\
&= \int_Y \left(\int_X \mathbf{1}_{D_y}(x) f(x)\, d\mu_1(x) \right) g(y)\, d\nu_1(y) \\
&= \int_Y \int_X \mathbf{1}_D(x, y) f(x) g(y)\, d\mu_1(x)\, d\nu_1(y) \\
&= \int_{X \times Y} \mathbf{1}_D(x, y) f(x) g(y)\, d(\mu_1 \times \nu_1)(x, y) \\
&= \int_D f(x) g(y)\, d(\mu_1 \times \nu_1)(x, y).
\end{aligned}$$