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Solutions Mid-term Exam Measure and Integration 2004

1. Let $\phi : [A, B] \to [a, b]$ be a strictly increasing surjective continuous function. Suppose $\psi : [a, b] \to \mathbb{R}$ is non-decreasing, and $f : [a, b] \to \mathbb{R}$ a bounded ψ-Riemann integrable function. Define α and g on $[A, B]$ by

$$\alpha(y) = \psi(\phi(y)) \text{ and } g(y) = f(\phi(y)).$$

Show that g is α-Riemann integrable, and

$$\int_A^B g d\alpha = \int_a^b f d\psi.$$

Proof: Since ϕ is strictly increasing surjective and continuous, then the inverse map ϕ^{-1} has the same properties. Hence, for any finite non-overlapping cover

$$\mathcal{C} = \{[a = a_0, a_1], \cdots, [a_{n-1}, a_n = b]\}$$

of $[a, b]$ corresponds a unique finite non-overlapping cover

$$\mathcal{C}' = \phi^{-1}(\mathcal{C}) = \{[A_0, A_1], \cdots, [A_{n-1}, A_n]\}$$

of $[A, B]$ such that $A_0 = A$, $B_0 = B$ and $A_i = \phi^{-1}(a_i)$. Conversely, with any finite non-overlapping cover

$$\mathcal{C}' = \{[A = A_0, A_1], \cdots, [A_{n-1}, A_n = B]\}$$

of $[A, B]$ corresponds a unique finite non-overlapping cover

$$\mathcal{C} = \phi(\mathcal{C}') = \{[a_0, a_1], \cdots, [a_{n-1}, a_n]\}$$

of $[a, b]$ such that $a_0 = a$, $b_0 = b$ and $a_i = \phi(A_i)$. Furthermore, $\mathcal{U}(g|\alpha; \mathcal{C}') = \mathcal{U}(f|\psi; \phi(\mathcal{C}'))$ and $\mathcal{L}(g|\alpha; \mathcal{C}') = \mathcal{L}(f|\psi; \phi(\mathcal{C}'))$.

Let $\epsilon > 0$, since f is ψ-Riemann integrable there exists a $\delta > 0$ such that if $\mathcal{C}$ is a finite non-overlapping cover of $[a, b]$ with $||\mathcal{C}|| < \delta$, then

$$\mathcal{U}(f|\psi; \mathcal{C}) - \mathcal{L}(f|\psi; \mathcal{C}) < \epsilon.$$

Thus, for any finite non-overlapping cover $\mathcal{C}'$ of $[A, B]$ such that $||\phi(\mathcal{C}')|| < \delta$ one has

$$\mathcal{U}(g|\alpha; \mathcal{C}') - \mathcal{L}(g|\alpha; \mathcal{C}') = \mathcal{U}(f|\psi; \mathcal{C}) - \mathcal{L}(f|\psi; \mathcal{C}) < \epsilon.$$

Thus,

$$\inf_{\mathcal{C}'} \mathcal{U}(g|\alpha; \mathcal{C}') - \sup_{\mathcal{C}'} \mathcal{L}(g|\alpha; \mathcal{C}') < \epsilon.$$

Therefore, g is α-Riemann integrable. Since,

$$\inf_{\mathcal{C}'} \mathcal{U}(g|\alpha; \mathcal{C}') = \inf_{\mathcal{C}} \mathcal{U}(f|\psi; \mathcal{C})$$

it follows that

$$\int_A^B g d\alpha = \int_a^b f d\psi.$$

2. Let $\{c_n\}$ be a sequence satisfying $c_n \geq 0$ for all $n \geq 1$, and $\sum_{n=1}^\infty c_n < \infty$. Let $\{s_n\}$ be a sequence of distinct points in (a, b). Define a function ψ on $[a, b]$ by $\psi(x) = \sum_{n=1}^\infty c_n 1_{(s_n,b]}(x)$, where $1_{(s_n,b]}$ is the indicator function of the interval $(s_n, b]$. Prove that any continuous function f on $[a, b]$ is ψ-Riemann integrable, and

$$\int_a^b f(x) d\psi(x) = \sum_{n=1}^\infty c_n f(s_n).$$

Proof: Clearly ψ is non-decreasing. Assume with no loss of generality that $s_1 < s_2 < s_3 < \cdots$, and let f be continuous on $[a, b]$. By Theorem 1.2.10, f is ψ-Riemann integrable. We now show that $\int_a^b f(x)d\psi(x) = \sum_{n=1}^\infty c_n f(s_n)$. Let $\epsilon > 0$, there exists a positive integer N such that $\sum_{n=m}^\infty c_n < \epsilon$ for all $m \geq N$. Choose any $m \geq N$, let $\psi_1(x) = \sum_{n=1}^m c_n 1_{(s_n,b]}(x)$ and $\psi_2(x) = \sum_{n=m+1}^\infty c_n 1_{(s_n,b]}(x)$. Then, f is ψ_1 and ψ_2-Riemann integrable, and

$$\int_a^b f(x)d\psi(x) = \int_a^b f(x)d\psi_1(x) + \int_a^b f(x)d\psi_2(x).$$

Notice that ψ_1 is constant on the intervals $[a, s_1], (s_1, s_2], \cdots, (s_m, b]$ with values $0, c_1, c_1 + c_2, \cdots, c_1 + c_2 + \cdots + c_m$ respectively. Thus by problem 2 in Exercises 2,

$$\int_a^b f(x)d\psi_1(x) = \sum_{n=1}^m c_n f(s_n).$$

Now, $\psi_2(b) = \sum_{n=m+1}^\infty c_n < \epsilon$ and $\psi_2(a) = 0$, thus by Theorem 1.2.10,

$$|\int_a^b f(x)d\psi_2(x)| \leq ||f||_u(\psi_2(b) - \psi_2(a)) \leq ||f||_u \epsilon.$$

Therefore, for each $m \geq N$,

$$|\int_a^b f(x)d\psi(x) - \sum_{n=1}^m c_n f(s_n)| = |\int_a^b f(x)d\psi_2(x)| \leq ||f||_u \epsilon.$$

This implies that

$$\int_a^b f(x)d\psi(x) = \sum_{n=1}^\infty c_n f(s_n).$$

3. Let $\Gamma \subseteq \mathbb{R}^n$. Recall that the inner Lebesgue measure of Γ is defined by

$$|\Gamma|_i = \sup\{|K| : K \subseteq \Gamma,\ K \text{ is compact}\}.$$

Prove the following.

(a) Assume $|\Gamma|_e < \infty$, then Γ is Lebesgue measurable **if and only if** $|\Gamma|_e = |\Gamma|_i$.

(b) Assume $|\Gamma|_e < \infty$, then Γ is Lebesgue measurable **if and only if** $|A|_e = |\Gamma \cap A|_e + |\Gamma^c \cap A|_e$ for all $A \subseteq \mathbb{R}^n$.

(c) If $A \subseteq \Gamma$, and Γ is Lebesgue measurable, then $|A|_e + |\Gamma \setminus A|_i = |\Gamma|$.

Proof (a): Suppose Γ is Lebesgue measurable, in this part we don't need the finiteness of $\Gamma|_e$. By problem 3 in Exercises 4 we have $|\Gamma|_i \leq |\Gamma|_e = |\Gamma|$. We will show that $|\Gamma| \leq |\Gamma|_i$. Let $\epsilon > 0$, since Γ^c is measurable, there exists an open set G such that $\Gamma^c \subseteq G$ and $|G \setminus \Gamma^c| < \epsilon$. Let $F = G^c$, then F is closed, $F \subseteq \Gamma$ and $|\Gamma \setminus F| = |G \setminus \Gamma^c| < \epsilon$. Let $K_n = F \cap \overline{B(0,n)}$ for $n \geq 1$. Then, $\{K_n\}$ is an increasing sequence of compact sets such that $F = \bigcup_{n=1}^{\infty} K_n$. Hence, $|F| = \lim_{n\to\infty} |K_n|$. If $|F| = \infty$, then $|\Gamma| = |\Gamma|_i = \infty$. Assume $|F| < \infty$. Then, there exists a positive integer N such that $|F| \leq |K_n| + \epsilon$ for all $n \geq N$. Let $n \geq N$, then

$$|\Gamma| \leq |F| + |\Gamma \setminus F| < |F| + \epsilon \leq |K_n| + 2\epsilon \leq |\Gamma|_i + 2\epsilon.$$

Since $\epsilon > 0$ is arbitrary, it follows that $|\Gamma| \leq |\Gamma|_i$. Therefore, $|\Gamma|_e = |\Gamma|_i$.

Conversely, suppose $|\Gamma|_e = |\Gamma|_i < \infty$. Let $\epsilon > 0$, then there exist a compact set K and an open set G such $K \subseteq \Gamma \subseteq G$, $|K| \geq |\Gamma|_e - \epsilon$ and $|G| \leq |\Gamma|_e + \epsilon$. Since K is compact, then $|K| < \infty$. Hence, $|G \setminus \Gamma|_e \leq |G \setminus K| = |G| - |K| \leq 2\epsilon$. Therefore, Γ is Lebesgue measurable.

Proof (b): Suppose Γ is Lebesgue measurable (we do not need finiteness of $|\Gamma|_e$), and let A be any subset of $\mathbb{R}^n$. By subadditivity of the outer Lebesgue measure, we have $|A|_e \leq |\Gamma \cap A|_e + |\Gamma^c \cap A|_e$. We prove the reverse inequality. Since Γ is Lebesgue measurable, for any open set G containing A, one has

$$|G| = |G \cap \Gamma| + |G \cap \Gamma^c| \geq |A \cap \Gamma|_e + |A \cap \Gamma^c|_e.$$

Thus,

$$|A|_e = \inf\{|G| : A \subseteq G, G \text{ open}\} \geq |A \cap \Gamma|_e + |A \cap \Gamma^c|_e.$$

Conversely, assume $|\Gamma|_e < \infty$, and suppose $|A|_e = |\Gamma \cap A|_e + |\Gamma^c \cap A|_e$ for all $A \subseteq \mathbb{R}^n$. By the hypothesis, for any open set G containing Γ, one has $|G| = |G \cap \Gamma|_e + |G \cap \Gamma^c|_e = |\Gamma|_e + |G \setminus \Gamma|_e$. Since $|\Gamma|_e < \infty$, then $|G \setminus \Gamma|_e = |G| - |\Gamma|_e$. Let $\epsilon > 0$, there exists an open set G containing Γ such that $|G| < |\Gamma|_e + \epsilon$, then $|G \setminus \Gamma|_e < \epsilon$. Thus, Γ is measurable.

Proof (c): For any open set G containing A,

$$|G|_e + |\Gamma \setminus A|_i \geq |G \cap \Gamma|_e + |\Gamma \setminus G|_i = |G \cap \Gamma| + |\Gamma \setminus G| = |\Gamma|.$$

Taking the infimum over open sets G containing A, we get $|A|_e + |\Gamma \setminus A|_i \geq |\Gamma|$.

Now, for any compact set $K \subseteq \Gamma \setminus A$,

$$|A|_e + |K| \leq |\Gamma \setminus K|_e + |K| = |\Gamma \setminus K| + |K| = |\Gamma|.$$

Taking the supremum over compact subsets K of $\Gamma \setminus A$, we get $|A|_e + |\Gamma \setminus A|_i \leq |\Gamma|$. Thus, $|A|_e + |\Gamma \setminus A|_i = |\Gamma|$.

4. Let E be a set, and $\mathcal{A}$ an algebra over E. Let $\mu : \mathcal{A} \to [0,1]$ be a function satisfying

(I) $\mu(E) = 1 = 1 - \mu(\emptyset)$,

(II) if $A_1, A_2, \cdots, \in \mathcal{A}$ are pairwise disjoint and $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$, then

$$\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n).$$

(a) Show that if $\{A_n\}$ and $\{B_n\}$ are increasing sequences in $\mathcal{A}$ such that $\bigcup_{n=1}^{\infty} A_n \subseteq \bigcup_{n=1}^{\infty} B_n$, then $\lim_{n\to\infty} \mu(A_n) \leq \lim_{n\to\infty} \mu(B_n)$.

(b) Let $\mathcal{G}$ be the collection of all subsets G of E such that there exists an increasing sequence $\{A_n\}$ in $\mathcal{A}$ with $G = \bigcup_{n=1}^{\infty} A_n$. Define $\overline{\mu}$ on $\mathcal{G}$ by

$$\overline{\mu}(G) = \lim_{n\to\infty} \mu(A_n),$$

where $\{A_n\}$ is an increasing sequence in $\mathcal{A}$ such that $G = \bigcup_{n=1}^{\infty} A_n$. Show the following.

(i) $\overline{\mu}$ is well defined.

(ii) If $G_1, G_2 \in \mathcal{G}$, then $G_1 \cup G_2, G_1 \cap G_2 \in \mathcal{G}$ and

$$\overline{\mu}(G_1 \cup G_2) + \overline{\mu}(G_1 \cap G_2) = \overline{\mu}(G_1) + \overline{\mu}(G_2).$$

(iii) If $G_n \in \mathcal{G}$ and $G_1 \subseteq G_2 \subseteq \cdots$, then $\bigcup_{n=1}^{\infty} G_n \in \mathcal{G}$ and

$$\overline{\mu}(\bigcup_{n=1}^{\infty} G_n) = \lim_{n\to\infty} \overline{\mu}(G_n).$$

(c) Define μ^* on $\mathcal{P}(E)$ (the power set of E) by

$$\mu^*(A) = \inf\{\overline{\mu}(G) : A \subseteq G,\ G \in \mathcal{G}\}.$$

(i) Show that $\mu^*(G) = \overline{\mu}(G)$ for all $G \in \mathcal{G}$, and

$$\mu^*(A \cup B) + \mu^*(A \cap B) \leq \mu^*(A) + \mu^*(B)$$

for all subsets A, B of E. Conclude that $\mu^*(A) + \mu^*(A^c) \geq 1$ for all $A \subseteq E$.

(ii) Show that if $C_1 \subseteq C_2 \subseteq \cdots$ are subsets of E and $C = \bigcup_{n=1}^{\infty} C_n$, then $\mu^*(C) = \lim_{n\to\infty} \mu^*(C_n)$.

(iii) Let $\mathcal{H} = \{B \subseteq E : \mu^*(B) + \mu^*(B^c) = 1\}$. Show that $\mathcal{H}$ is a σ-algebra over E, and μ^* is a measure on $\mathcal{H}$.

(iv) Show that $\sigma(E; \mathcal{A}) \subseteq \mathcal{H}$. Conclude that the restriction of μ^* to $\sigma(E; \mathcal{A})$ is a measure extending μ, i.e. $\mu^*(A) = \mu(A)$ for all $A \in \mathcal{A}$.

Proof (a): Using the same proof as in Theorem 3.1.6 (i), one can easily show that if $\{D_n\}$ is an increasing sequence in $\mathcal{A}$ such that $\bigcup_n D_n \in \mathcal{A}$, then $\mu(\bigcup_n D_n) = \lim_{n\to\infty} \mu(D_n)$. Suppose that $\{A_n\}$ and $\{B_n\}$ are increasing sequences in $\mathcal{A}$ such that $\bigcup_{n=1}^{\infty} A_n \subseteq \bigcup_{n=1}^{\infty} B_n$. For each $m \geq 1$, $\{A_m \cap B_n : n \geq 1\}$ is an increasing sequence in $\mathcal{A}$ and $A_m = A_m \cap \bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty}(A_m \cap B_n) \in \mathcal{A}$. Thus, for each $m \geq 1$,

$$\mu(A_m) = \lim_{n\to\infty} \mu(A_m \cap B_n) \leq \lim_{n\to\infty} \mu(B_n).$$

Taking the limit as $m \to \infty$, we get $\lim_{m\to\infty} \mu(A_m) \leq \lim_{n\to\infty} \mu(B_n)$.

Proof (b)(i): Let $G \in \mathcal{G}$. If $\{A_n\}$ and $\{B_n\}$ are two increasing sequences in $\mathcal{A}$ such that $G = \bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n$. Then, by part (a) $\lim_{m\to\infty} \mu(A_m) = \lim_{n\to\infty} \mu(B_n)$. This shows that $\overline{\mu}$ is well defined on $\mathcal{G}$.

Proof (b)(ii): Let $G_1, G_2 \in \mathcal{G}$, there exist increasing sequences $\{A_n\}$, $\{B_n\}$ in $\mathcal{A}$ such that $G_1 = \bigcup_{n=1}^{\infty} A_n$ and $G_1 = \bigcup_{n=1}^{\infty} B_n$. Then, $\{A_n \cup B_n\}$, $\{A_n \cap B_n\}$ are increasing sequences in $\mathcal{G}$ such that $G_1 \cup G_2 = \bigcup_{n=1}^{\infty}(A_n \cup B_n)$ and $G_1 \cap G_2 = \bigcup_{n=1}^{\infty}(A_n \cap B_n)$. Thus, $G_1 \cup G_2, G_1 \cap G_2 \in \mathcal{G}$. By definition of $\overline{\mu}$,

$$\begin{aligned}
\overline{\mu}(G_1 \cup G_2) &= \lim_{n\to\infty} \mu(A_n \cup B_n) \\
&= \lim_{n\to\infty} (\mu(A_n) + \mu(B_n) - \mu(A_n \cap B_n)) \\
&= \overline{\mu}(G_1) + \overline{\mu}(G_2) - \overline{\mu}(G_1 \cap G_2).
\end{aligned}$$

Proof (b)(iii): For each $n \geq 1$ there exists an increasing sequence $\{A_{nm} : m \geq 1\}$ in $\mathcal{A}$ such that $G_n = \bigcup_{m=1}^{\infty} A_{nm}$. Let $D_m = \bigcup_{n=1}^{m} A_{nm}$ for $m \geq 1$, then $\{D_m\}$ is an increasing sequence in $\mathcal{A}$. For each $n \leq m$, $A_{nm} \subseteq D_m \subseteq G_m$. and hence $\mu(A_{nm}) \leq \mu(D_m) \leq \overline{\mu}(G_m)$. We will show that $\bigcup_{n=1}^{\infty} G_n = \bigcup_{n=1}^{\infty} D_n$.

For any $n \geq 1$,

$$G_n = \bigcup_{m=1}^{\infty} A_{nm} = \bigcup_{m=n}^{\infty} A_{nm} \subseteq \bigcup_{m=1}^{\infty} D_m \subseteq \bigcup_{m=1}^{\infty} G_m.$$

Thus,

$$\bigcup_{n=1}^{\infty} G_n \subseteq \bigcup_{m=1}^{\infty} D_m \subseteq \bigcup_{m=1}^{\infty} G_m.$$

Hence, $\bigcup_{n=1}^{\infty} G_n = \bigcup_{n=1}^{\infty} D_n$, and $\bigcup_{n=1}^{\infty} G_n \in \mathcal{G}$. From $\mu(A_{nm}) \leq \mu(D_m) \leq \overline{\mu}(G_m)$, $n \leq m$ one gets for each $n \geq 1$,

$$\overline{\mu}(G_n) = \lim_{m\to\infty} \mu(A_{nm}) \leq \lim_{m\to\infty} \mu(D_m) = \overline{\mu}(\bigcup_{n=1}^{\infty} G_n) \leq \lim_{m\to\infty} \overline{\mu}(G_m).$$

Taking the limit as $n \to \infty$, we get

$$\lim_{n\to\infty} \overline{\mu}(G_n) = \overline{\mu}(\bigcup_{n=1}^{\infty} G_n).$$

Proof (c)(i): Let $G \in \mathcal{G}$, by definition of μ^*, $\mu^*(G) \leq \overline{\mu}(G)$. Notice that part(a) implies that $\overline{\mu}$ is monotone. Hence, for any $G' \in \mathcal{G}$ containing G we have $\overline{\mu}(G) \leq \overline{\mu}(G')$. Taking the infimum over all sets $G' \in \mathcal{G}$ containing G we get $\overline{\mu}(G) \leq \mu^*(G)$.

Now, let A, B be any two subsets of E and let $\epsilon > 0$. There exist sets $G_1, G_2 \in \mathcal{G}$ such that $\overline{\mu}(G_1) \leq \mu^*(A)+\epsilon$, and $\overline{\mu}(G_2) \leq \mu^*(B)+\epsilon$. By part (b)(ii), $A \cap B \subseteq G_1 \cap G_2 \in \mathcal{G}$ and $A \cup B \subseteq G_1 \cup G_2 \in \mathcal{G}$, hence

$$\mu^*(A \cup B)+\mu^*(A \cap B) \leq \overline{\mu}(G_1 \cup G_2)+\overline{\mu}(G_1 \cap G_2) = \overline{\mu}(G_1)+\overline{\mu}(G_2) \leq \mu^*(A)+\mu^*(B)+2\epsilon.$$

Since $\epsilon > 0$ is arbitrary, it follows that $\mu^*(A \cup B)+\mu^*(A \cap B) \leq \mu^*(A)+\mu^*(B)$. Finally, taking $B = A^c$ and noticing that $\mu^*(E) = 1 = 1-\mu^*(\emptyset)$, we get $1 \leq \mu^*(A)+\mu^*(A^c)$ for all $A \subseteq E$.

Proof (c)(ii): Let $\{C_n\}$ be an increasing sequence of subsets of E and let $C = \bigcup_{n=1}^{\infty} C_n$. Since μ^* is clearly monotone, it follows that $\mu^*(C_n) \leq \mu^*(C)$ for all $n \geq 1$. Hence, $\lim_{n\to\infty} \mu^*(C_n) \leq \mu^*(C)$. We now prove the reverse inequality. Let $\epsilon > 0$, for each n choose $G_n \in \mathcal{G}$ such that $\overline{\mu}(G_n) \leq \mu^*(C_n) + \dfrac{\epsilon}{2^n}$. Let $G = \bigcup_{n=1}^{\infty} G_n$ and $F_n = \bigcup_{m=1}^{n} G_n$. Then, $C \subseteq G$, $\{F_n\}$ is an increasing sequence in $\mathcal{G}$ and $G = \bigcup_{n=1}^{\infty} F_n$. By part (b)(iii), $G \in \mathcal{G}$ and $\mu^*(C) \leq \mu^*(G) = \lim_{n\to\infty} \mu^*(F_n)$. Finally, using induction, one can easily show that $\mu^*(F_n) = \overline{\mu}(F_n) \leq \mu^*(C_n) + \sum_{i=1}^{n} \dfrac{\epsilon}{2^i}$. From this it follows that

$$\mu^*(C) \leq \lim_{m\to\infty} \mu^*(F_n) \leq \lim_{n\to\infty} \mu^*(C_n) + \epsilon.$$

Thus, $\mu^*(C) \leq \lim_{n\to\infty} \mu^*(C_n)$.

Proof (c)(iii): Clearly, $\emptyset \in \mathcal{H}$ and $\mathcal{H}$ is closed under complementation. We first show that $\mathcal{H}$ is an algebra. Let $H_1, H_2 \in \mathcal{H}$. By part (c)(i),

$$\mu^*(H_1 \cup H_2) + \mu^*(H_1 \cap H_2) \leq \mu^*(H_1) + \mu^*(H_2)$$

and

$$\mu^*((H_1 \cup H_2)^c) + \mu^*((H_1 \cap H_2)^c) \leq \mu^*(H_1^c) + \mu^*(H_2^c).$$

Adding both equations, and using that $H_1, H_2 \in \mathcal{H}$ and the last conclusion of part (b)(i), we get

$$2 \leq \mu^*(H_1 \cup H_2) + \mu^*((H_1 \cup H_2)^c) + \mu^*(H_1 \cap H_2) + \mu^*((H_1 \cap H_2)^c) = 2.$$

Since, $\mu^*(H_1 \cup H_2)+\mu^*((H_1 \cup H_2)^c) \geq 1$ and $\mu^*(H_1 \cap H_2)+\mu^*((H_1 \cap H_2)^c) \geq 1$, we must have that $\mu^*(H_1 \cup H_2) + \mu^*((H_1 \cup H_2)^c) = 1$ and $\mu^*(H_1 \cap H_2) + \mu^*((H_1 \cap H_2)^c) = 1$. Thus, $H_1 \cup H_2, H_1 \cap H_2 \in \mathcal{H}$ and $\mathcal{H}$ is an algebra. Furthermore, from the above anlaysis we must have $\mu^*(H_1 \cup H_2) + \mu^*(H_1 \cap H_2) = \mu^*(H_1) + \mu^*(H_2)$ otherwise the sum of the first two displayed equations would be less than 2, a contradiction. Thus, μ^* is additive on $\mathcal{H}$.

We now show that $\mathcal{H}$ is a σ-algebra. Let $H_1, H_2, \cdots, \in \mathcal{H}$ and let $H = \bigcup_{n=1}^{\infty} H_n$. To show that $H \in \mathcal{H}$, it is enough to show that $\mu^*(H) + \mu^*(H^c) \leq 1$ (see part (c)(i)). Let $G_n = \bigcup_{m=1}^{n} H_m$. Since $\mathcal{H}$ is an algebra, then $\{G_n\}$ is an increasing sequence in $\mathcal{H}$ such that $H = \bigcup_{n=1}^{\infty} G_n$. Hence, by part (c)(ii), $\mu^*(H) = \lim_{n\to\infty} \mu^*(G_n)$. Let $\epsilon > 0$, there exists a positive integer N such that $\mu^*(H) \leq \mu^*(G_n) + \epsilon$ for all $n \geq N$. Now, $H^c \subseteq G_n^c$, hence $\mu^*(H) \leq \mu^*(G_n^c)$ for all $n \geq 1$. For any $n \geq N$, we have

$$\mu^*(H) + \mu^*(H^c) \leq \mu^*(G_n) + \mu^*(G_n^c) + \epsilon = 1 + \epsilon.$$

Since, $\epsilon > 0$ is arbitrary, it follows that $\mu^*(H)+\mu^*(H^c) \leq 1$. Thus, $H \in \mathcal{H}$, and $\mathcal{H}$ is σ-algebra. Finally, we show that μ^* is σ-additive on $\mathcal{H}$. Let $H_1, H_2, \cdots, \in \mathcal{H}$ be pairwise disjoint, and let $G_n = G_n = \bigcup_{m=1}^{n} H_m$. Then, $\{G_n\}$ is an increasing sequence in $\mathcal{H}$ such that $\bigcup_{n=1}^{\infty} H_n = \bigcup_{n=1}^{\infty} G_n$. By part (c)(ii) and the (finite) additivity of μ^* on $\mathcal{H}$, we get

$$\mu^*(\bigcup_{n=1}^{\infty} H_n) = \lim_{n\to\infty} \mu^*(G_n) = \lim_{n\to\infty} \sum_{m=1}^{n} \mu^*(H_m) = \sum_{m=1}^{\infty} \mu^*(H_m).$$

Thus, μ^* is a measure on $\mathcal{H}$.

Proof (c)(iv): Since $\mathcal{A} \subseteq \mathcal{G}$, it is enough to show that $\mathcal{G} \subseteq \mathcal{H}$. Let $G \in \mathcal{G}$, and $\{A_n\}$ an increasing sequence in $\mathcal{A}$ such that $G = \bigcup_{n=1}^{\infty} A_n$. By part (b), $\overline{\mu}(G) = \mu^*(G) = \lim_{n\to\infty} \mu(A_n)$. Notice that for each $n \geq 1$, $\mu(A_n) = \overline{\mu}(A_n) = \mu^*(A_n)$, and $G^c \subseteq A_n^c$. Thus, for each $n \geq 1$,

$$\mu(A_n) + \mu^*(G^c) \leq \mu(A_n) + \mu(A_n^c) = 1.$$

Taking the limit as $n \to \infty$ we get,

$$\mu^*(G) + \mu^*(G^c) \leq 1.$$

By part (c)(i), this implies that $\mu^*(G) + \mu^*(G^c) = 1$, and hence $G \in \mathcal{H}$. Therefore, $\sigma(E; \mathcal{A}) \subseteq \mathcal{H}$ and the restriction of μ^* to $\sigma(E; \mathcal{A})$ is a measure extending μ.

5. Let $\overline{\mathcal{B}}_{\mathbb{R}^N}$ be the Lebesgue σ-algebra over $\mathbb{R}^N$, $\mathcal{B}_{\mathbb{R}^N}$ the Borel σ-algebra over $\mathbb{R}^N$, and $\mathcal{B}_{\overline{\mathbb{R}}}$ the Borel σ-algebra over $\overline{\mathbb{R}} = [-\infty, \infty]$. Denote by $\lambda_{\mathbb{R}^N}$ the Lebesgue measure on $\overline{\mathcal{B}}_{\mathbb{R}^N}$. Let $f : \mathbb{R}^N \to [-\infty, \infty]$ be Lebesgue measurable (i.e. $f^{-1}(A) \in \overline{\mathcal{B}}_{\mathbb{R}^N}$ for all $A \in \mathcal{B}_{\overline{\mathbb{R}}}$). Show that there exists a function $g : \mathbb{R}^N \to [-\infty, \infty]$ which is Borel measurable (i.e. $g^{-1}(A) \in \mathcal{B}_{\mathbb{R}^N}$ for all $A \in \mathcal{B}_{\overline{\mathbb{R}}}$) such that

$$\lambda_{\mathbb{R}^N}\left(\{x \in \mathbb{R}^N : f(x) \neq g(x)\}\right) = 0.$$

(Hint: assume first that f is a non-negative simple function)

Proof: Assume first that f is a non-negative Lebesgue measurable simple function. Then f has the form $f = \sum_{i=1}^{n} a_i 1_{A_i}$, where $a_1, a_2, \cdots, a_n$ are all distinct, and $A_1, A_2, \cdots, A_n \in \overline{\mathcal{B}}_{\mathbb{R}^N}$ are pairwise disjoint. Since every Lebesgue set is the disjoint union of a Borel Set and Lebesgue set of Lebesgue measure zero, it follows that for each $i = 1, 2, \cdots, n$ $A_i = B_i \cup N_i$, where $B_i \in \mathcal{B}_{\overline{\mathbb{R}}}$, and $\lambda_{\mathbb{R}^N}(N_i) = 0$. Let $g = \sum_{i=1}^{n} a_i 1_{B_i}$, then g is Borel measurable, and $\lambda_{\mathbb{R}^N}\left(\{x \in \mathbb{R}^N : f(x) \neq g(x)\}\right) \leq \lambda_{\mathbb{R}^N}(\cup_{i=1}^{n} N_i) = 0$. Now assume f is a non-negative Lebesgue measurable function. Then there exists an increasing sequence $\{\phi_n\}$ of non-negative Lebesgue measurable simple functions such that $f = \lim_{n\to\infty} \phi_n = \sup_n \phi_n$. Each ϕ_n has the form $\phi_n = \sum_{i=1}^{m_n} a_i^{(n)} 1_{A_i^{(n)}}$, where $a_i^{(n)}$ are all distinct and $A_i^{(n)} \in \overline{\mathcal{B}}_{\mathbb{R}^N}$. Further, $A_i^{(n)} = B_i^{(n)} \cup N_i^{(n)}$ (disjoint union), where $B_i^{(n)} \in \mathcal{B}_{\overline{\mathbb{R}}}$ and $\lambda_{\mathbb{R}^N}(N_i^{(n)}) = 0$. Set $g_n = \sum_{i=1}^{m_n} a_i^{(n)} 1_{B_i^{(n)}}$, then g_n is Borel measurable, $0 \leq g_n \leq \phi_n$ and $\lambda_{\mathbb{R}^N}(\phi_n \neq g_n) \leq \lambda_{\mathbb{R}^N}(\cup_{i=1}^{m_n} N_i^{(n)}) = 0$. Let $g = \sup_n g_n$. Then g is Borel measurable, $0 \leq g \leq f$ and $\lambda_{\mathbb{R}^N}(f \neq g) \leq$

$\lambda_{\mathbb{R}^N}(\cup_{n=1}^{\infty}\cup_{i=1}^{m_n} N_i^{(n)}) = 0$. Finally, let f be any Lebesgue measurable function. Then $f = f^+ - f^-$ with f^+, f^- non-negative Lebesgue measurable functions. By the above, there exist h_1, h_2 Borel measurable such that $0 \le h_1 \le f^+$, $0 \le h_2 \le f^-$, and $\lambda_{\mathbb{R}^N}(f^+ \ne h_1) = \lambda_{\mathbb{R}^N}(f^- \ne h_2) = 0$. Then, $h_1 - h_2$ is a Borel measurable function (note that $h_1 - h_2$ has never the value $\infty - \infty$ since $0 \le h_1 \le f^+$ and $0 \le h_2 \le f^-$), and $\lambda_{\mathbb{R}^N}(f \ne h_1 - h_2) \le \lambda_{\mathbb{R}^N}(f^+ \ne h_1) + \lambda_{\mathbb{R}^N}(f^- \ne h_2) = 0$.

6. Let $(E, \mathcal{B}, \mu)$ be a measure space, and $f : E \to [0, \infty]$ a measurable **simple** function such that $\int_E f\, d\mu < \infty$. Show that for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $A \in \mathcal{B}$ with $\mu(A) < \delta$ then $\int_A f\, d\mu < \epsilon$.

   Proof: The proof is done by contradiction. Suppose there exists an $\epsilon > 0$ such that for every $\delta > 0$ there exists a measurable set A such that $\mu(A) < \delta$ but $\int_A f d\mu \ge \epsilon$. For $A \in \mathcal{B}$, let $\lambda(A) = \int_A f\, d\mu$. By problem 3 of Exercises 8, λ is a finite measure on $\mathcal{B}$. By our assumption, for each $n \ge 1$ there exists a measurable subset A_n such that $\mu(A_n) < \dfrac{1}{2^n}$ and $\lambda(A_n) = \int_{A_n} f d\mu \ge \epsilon$. Let $A = \limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty}\bigcup_{m=n}^{\infty} A_m$. Since $\sum_{n=1}^{\infty}\mu(A_n) < \infty$, then by Borel-Cantelli Lemma (problem 3(c) in Exercises 7) we have $\mu(A) = 0$. But then $\lambda(A) = \int_A f d\mu = 0$. Since λ is a finite measure, by problem 3(b) in Exercises 7, we have

$$0 = \lambda(A) = \lambda(\limsup_{n\to\infty} A_n) \ge \limsup_{n\to\infty} \lambda(A_n) \ge \epsilon,$$

   a contradiction. Therefore, for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $A \in \mathcal{B}$ with $\mu(A) < \delta$ then $\int_A f\, d\mu < \epsilon$.

   Note that in the proof we did not use the fact the f is a non-negative **simple** function, hence the proof holds for any non-negative measurable μ-integrable function on $(E, \mathcal{B}, \mu)$.