Universiteit Utrecht Universiteit Utrecht Boedapestlaan 6

Mathematisch Instituut 3584 CD Utrecht

Measure and Integration Exercises 11

1. Let $(E, \mathcal{B}, \mu)$ be a measure space, and $f_n : E \to \mathbb{R}$ a sequence of measurable real valued functions on $(E, \mathcal{B}, \mu)$.

   (a) Suppose $f : E \to \mathbb{R}$ is measurable. Show that

   $$\{x \in E : \lim_{n\to\infty} f_n(x) \neq f(x)\} = \bigcup_{l=1}^{\infty} \bigcap_{m=1}^{\infty} \{x \in E : \sup_{n\geq m} |f_n(x) - f(x)| \geq 1/l\}.$$

   (b) Show that if $f_n \to f$ μ a.e., then for every $\epsilon > 0$

   $$\mu(\bigcap_{m=1}^{\infty} \{x \in E : \sup_{n\geq m} |f_n(x) - f(x)| \geq \epsilon\}) = 0.$$

Proof (a) Let $B = \{x \in E : \lim_{n\to\infty} f_n(x) \neq f(x)\}$ and suppose $x \in B$. Then, there exists $\epsilon > 0$ such that for every $m \geq 1$, there exists an $n \geq m$ such that $|f_n(x) - f(x)| \geq \epsilon$. This implies that for each $m \geq 1$, one has $\sup\limits_{n\geq m} |f_n(x) - f(x)| \geq \epsilon$. Furtheremore, there exists $l \geq 1$ such that $1/l < \epsilon$, then for each $m \geq 1$, $\sup\limits_{n\geq m} |f_n(x) - f(x)| \geq \epsilon > 1/l$. Thus,

$$x \in \bigcap_{m=1}^{\infty} \{x \in E : \sup_{n\geq m} |f_n(x) - f(x)| \geq 1/l\} \subseteq \bigcup_{l=1}^{\infty} \bigcap_{m=1}^{\infty} \{x \in E : \sup_{n\geq m} |f_n(x) - f(x)| \geq 1/l\}.$$

Conversely, let $x \in \bigcup_{l=1}^{\infty} \bigcap_{m=1}^{\infty} \{x \in E : \sup_{n\geq m} |f_n(x) - f(x)| \geq 1/l\}$, then there exists $l \geq 1$ so that for all $m \geq 1$, $\sup\limits_{n\geq m} |f_n(x) - f(x)| \geq 1/l$. Then for any $0 < \epsilon < 1/l$, one has $\sup_{n\geq m} |f_n(x) - f(x)| > \epsilon$. In other words, for each $m \geq 1$ there exists $n \geq m$ such that $|f_n(x) - f(x)| \geq \epsilon$. Hence, $x \in B$.

Proof (b) If $f_n \to f$ μ a.e, then $\mu(B) = 0$. Hence, by part (a) one has $\mu(\bigcap_{m=1}^{\infty} \{x \in E : \sup_{n\geq m} |f_n(x) - f(x)| \geq 1/l\}) = 0$ for each $l \geq 1$. But for any $\epsilon > 0$, there exists an $l \geq 1$ such that $1/l < \epsilon$, then

$$\mu(\bigcap_{m=1}^{\infty} \{x \in E : \sup_{n\geq m} |f_n(x) - f(x)| \geq \epsilon\}) \leq \mu(\bigcap_{m=1}^{\infty} \{x \in E : \sup_{n\geq m} |f_n(x) - f(x)| \geq 1/l\}) = 0$$

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2. Consider the measure space $([0,1), \mathcal{B}_{[0,1)}, \lambda_{[0,1)})$, where $\mathcal{B}_{[0,1)}$ and $\lambda_{[0,1)}$ are the restrictions of the Borel σ-algebra and Lebesgue measure on $[0,1)$. Define a sequence of measurable functions f_n on $[0,1)$ as follows: given $n \geq 1$, there exist an $m \geq 0$ and $0 \leq l \leq 2^m - 1$ such that $n = 2^m + l$ (note that this representation is unique). Set $f_n = f_{2^m+l} = 1_{[l/2^m,(l+1)/2^m)}$.

   (a) Determine explicitly $f_1, f_2, f_3, f_4, f_5, f_6, f_7$.

   (b) Show that $\limsup_{n\to\infty} f_n(x) = 1$ for all $x \in [0,1]$.

   (c) Show that $\lim_{n\to\infty} ||f||_{L^1(\lambda_{[0,1]})} = 0$. Conclude that L^1-convergence does not imply μ a.e. convergence.

Proof (a) Notice that $1 = 2^0+0$, $2 = 2^1+0$, $3 = 2^1+1$, $4 = 2^2+0$, $5 = 2^2+1$, $6 = 2^2+2$, $7 = 2^2+3$. Thus, $f_1 = 1_{[0,1)}$, $f_2 = 1_{[0,1/2)}$, $f_3 = 1_{[1/2,1)}$, $f_4 = 1_{[0,1/4)}$, $f_5 = 1_{[1/4,2/4)}$, $f_6 = 1_{[2/4,3/4)}$, $f_7 = 1_{[3/4,1)}$.

Proof (b) Notice that for each $m \geq 1$, $\{[l/2^m, (l+1)/2^m) : 0 \leq l \leq 2^m - 1\}$ forms a partition of $[0,1)$. Hence, for each $x \in [0,1)$ and for every $m \geq 0$ there exists an $0 \leq l \leq 2^m - 1$ such that $f_{2^m+l}(x) = 1$. Thus $f_n(x) = 1$ for infinitely many n. This shows that $\limsup_{n\to\infty} f_n(x) = 1$ for all $x \in [0,1]$.

Proof (c) If $n = 2^m+l$, then $||f_n||_{L^1(\lambda_{[0,1)})} = \int_{[0,1)} f_{2^m+l} d\lambda_{[0,1)} = 1/2^m$. Since $m \to \infty$ as $n \to \infty$, taking limits we see that $\lim_{n\to\infty} ||f||_{L^1(\lambda_{[0,1)})} = 0$. This shows that $f_n \to 0$ in $L^1(\lambda_{[0,1)})$ but from part (b), f_n **does not** converge to 0 μ a.e.

3. Consider the measure space $([a,b], \mathcal{B}, \lambda)$, where $\mathcal{B}$ is the Borel σ-algebra on $[a,b]$, and λ is the restriction of the Lebesgue measure on $[a,b]$. Let $f : [a,b] \to \mathbb{R}$ be any continuous function. Show that the Riemann integral of f on $[a,b]$ is equal to the Lebesgue integral of f on $[a,b]$, i.e.

$$(R)\int_a^b f(x)dx = \int_{[a,b]} f d\lambda.$$

Proof Since f is continuous, then f is Riemann integrable on $[a,b]$. For each $n \geq 1$, divide the interval $[a,b]$ into 2^n intervals of equal length $I_0^{(n)}, I_1^{(n)}, \cdots, I_{2^n-1}^{(n)}$, where

$$I_j^{(n)} = \left[a + \frac{j(b-a)}{2^n}, a + \frac{(j+1)(b-a)}{2^n}\right].$$

Let $\mathcal{C}^{(n)} = \{I_j^{(n)} : 0 \leq j \leq 2^n - 1.\}$. Notice that $\mathcal{C}^{(n+1)}$ is a refinement of $\mathcal{C}^{(n)}$, and $||\mathcal{C}^{(n)}|| = \dfrac{1}{2^n} \to 0$ as $n \to \infty$. For each n, define the choice fuction $\xi^{(n)}$ by $\xi^{(n)}(I_j^{(n)}) = a + \frac{j(b-a)}{2^n}$. Then,

$$\mathcal{R}(f; \mathcal{C}^{(n)}, \xi^{(n)}) = \sum_{j=0}^{2^n-1} f(a + \frac{j(b-a)}{2^n}) \cdot \frac{1}{2^n}.$$

By Riemann integrability of f we have

$$\lim_{n\to\infty}\sum_{j=0}^{2^n-1} f(a+\frac{j(b-a)}{2^n})\cdot\frac{1}{2^n}=\lim_{n\to\infty}\mathcal{R}(f;\mathcal{C}^{(n)},\xi^{(n)})=(R)\int_a^b f(x)dx.$$

Now, let $f_n=\sum_{j=0}^{2^n-1} f(a+\frac{j(b-a)}{2^n})\cdot 1_{I_j^{(n)}}$, then $|f_n|\le ||f||_u$ for all n. By uniform continuity of f, given any $\epsilon>0$, there exists a $\delta>0$ such that $|f(x)-f(y)|<\epsilon$ whenever $|x-y|<\delta$, $x,y\in[a,b]$. Moreover, there exists an integer $N\ge 1$ such that $\dfrac{1}{2^n}<\delta$ for all $n\ge N$. Thus, if $n\ge N$, then $|f_n(x)-f(x)|<\epsilon$ for all $x\in[a,b]$. This implies that $f_n\to f$ pointwise on $[a,b]$. Since f is bounded, then $\int_{[a,b]}|f|d\lambda\le ||f||_u(b-a)$, and hence f is λ-integrable. By the Lebesgue Dominated Convergence Theorem,

$$\int_{[a,b]} fd\lambda=\lim_{n\to\infty}\int_{[a,b]} f_n d\lambda=\lim_{n\to\infty}\sum_{j=0}^{2^n-1} f(a+\frac{j(b-a)}{2^n})\cdot\frac{1}{2^n}=(R)\int_a^b f(x)dx.$$

4. Let $(E,\mathcal{B},\mu)$ be a measure space, and $f_n:E\to\mathbb{R}$ a sequence of measurable real valued functions on $(E,\mathcal{B},\mu)$. Let $f:E\to\mathbb{R}$ be a measurable function such that $\sum_{n=0}^{\infty}\mu(|f-f_n|\ge\epsilon))<\infty$ for all $\epsilon>0$. Show that $f_n\to f$ in μ-measure and μ a.e.

   Proof. Let $\epsilon>0$ be given. For any $0<\epsilon'<\epsilon$, and any integer $m\ge 1$,

$$\mu(\sup_{n\ge m}|f-f_n|\ge\epsilon)\le\mu\left(\bigcup_{n=m}^{\infty}\{|f-f_n|\}\ge\epsilon'\right)\le\sum_{n=m}^{\infty}\mu(|f-f_n|\ge\epsilon').$$

   Since $\lim_{m\to\infty}\sum_{n=m}^{\infty}\mu(|f-f_n|\ge\epsilon')=0$, it follows that

$$\lim_{m\to\infty}\mu(\sup_{n\ge m}|f-f_n|\ge\epsilon)=0.$$

   By Theorem 3.3.7, $f_n\to f$ in μ-measure and μ a.e.