Universiteit Utrecht Universiteit Utrecht Boedapestlaan 6

Mathematisch Instituut 3584 CD Utrecht

Measure and Integration Exercises 13

1. Let $(E_1, \mathcal{B}_1, \mu_1)$ and $(E_2, \mathcal{B}_2, \mu_2)$ be finite measure spaces. Let $\Gamma \in \mathcal{B}_1 \times \mathcal{B}_2$. For $x_1 \in E_1$, $x_2 \in E_2$, let $\Gamma(x_1) = \{x_2 \in E_2 : (x_1, x_2) \in \Gamma\}$ and $\Gamma(x_2) = \{x_1 \in E_1 : (x_1, x_2) \in \Gamma\}$. Let

$$\mathcal{D} = \{\Gamma \in \mathcal{B}_1 \times \mathcal{B}_2 : \mu_2(\Gamma(x_1)) \text{ and } \mu_1(\Gamma(x_2)) \text{ are measurable } \}.$$

Show that $\mathcal{D}$ is a λ-system containing $\mathcal{C} = \{\Gamma_1 \times \Gamma_2 : \Gamma_i \in \mathcal{B}_i, i = 1, 2\}$. Conclude that $\mathcal{D} = \mathcal{B}_1 \times \mathcal{B}_2$.

Proof To show $\mathcal{D}$ is a λ-system, we need to check conditions (a)-(d) on page 34. First notice that if $C, D \in \mathcal{B}_1 \times \mathcal{B}_2$, and $x_1 \in E_1$, then $D^c(x_1) = (D(x_1))^c$, $(C \cup D)(x_1) = C(x_1) \cup D(x_1)$, and $(D \setminus C)(x_1) = D(x_1) \setminus C(x_1)$. Similarly, if we replace x_1 by x_2. From this conditions (a) and (b) easily follow. Now suppose $C, D \in \mathcal{D}$ with $C \subseteq D$, then $\mu_2((D \setminus C)(x_1)) = \mu_2(D(x_1)) - \mu_2(C(x_1))$ is measurable (notice that the difference is well defined since μ_2 is a finite measure). Similarly, $\mu_1((D \setminus C)(x_1)) = \mu_2(D(x_1)) - \mu_2(C(x_1))$. Now, let $D_1 \subseteq D_2 \subseteq \cdots$ be a sequence in $\mathcal{D}$, then for any $x_1 \in E_1$ and $x_2 \in E_2$, one has that $(\mu_2(D_n(x_1)))$ and $(\mu_1(D_n(x_2)))$ are increasing sequences of non-negative measurable functions on $((E_1, \mathcal{B}_1)$ and $((E_2, \mathcal{B}_2)$ respectively. By Monotone convergence theorem, we have that $\lim\limits_{n\to\infty}(\mu_2(D_n(x_1)))$ and $\lim\limits_{n\to\infty}(\mu_1(D_n(x_2)))$ are measurable. But by theorem 3.1.6 (p. 36),

$$\mu_2((\bigcup_{n=1}^{\infty} D_n)(x_1)) = \mu_2(\bigcup_{n=1}^{\infty}(D_n)(x_1)) = \lim_{n\to\infty}(\mu_2(D_n(x_1)))$$

and

$$\mu_1((\bigcup_{n=1}^{\infty} D_n)(x_2)) = \mu_1(\bigcup_{n=1}^{\infty}(D_n)(x_2)) = \lim_{n\to\infty}(\mu_1(D_n(x_2))).$$

This shows that $\bigcup_{n=1}^{\infty} D_n \in \mathcal{D}$. Thus, $\mathcal{D}$ is a λ-system. To show $\mathcal{C} \subset \mathcal{D}$, notice that if $\Gamma_i \in \mathcal{B}_i$, then $(\Gamma_1 \times \Gamma_2)(x_1) = \Gamma_2$ and $(\Gamma_1 \times \Gamma_2)(x_2) = \Gamma_1$. Thus, $\mu_2((\Gamma_1 \times \Gamma_2)(x_1)) = \mu_2(\Gamma_2)$ and $\mu_1((\Gamma_1 \times \Gamma_2)(x_2)) = \mu_1(\Gamma_1)$ are constants, hence measurable. Therefore, $\Gamma_1 \times \Gamma_2 \in \mathcal{D}$. By Lemma 3.1.3 (p. 34), we have $\mathcal{D} = \mathcal{B}_1 \times \mathcal{B}_2$.

2. Suppsose that μ is σ-finite on $(E, \mathcal{B})$, and write $E = \bigcup_{n=1}^{\infty} E_n$, where E_n are measurable, pairwise disjoint and $\mu(E_n) < \infty$. Define μ_n on $\mathcal{B}$ by $\mu_n(A) = \mu(A \cap E_n)$. Show that μ_n is a measure on $(E, \mathcal{B})$ and for every $f \geq 0$ measurable,

$$\int_E f\, d\mu = \sum_{n=1}^{\infty} \int_{E_n} f\, d\mu_n.$$

Proof The proof that μ_n is a measure is left to the reader. We prove the second statement. Suppose first that $f = 1_A$, where $A \in \mathcal{B}$. Notice that

$$\mu(A) = \sum_{n=1}^{\infty} \mu(A \cap E_n) = \sum_{n=1}^{\infty} \mu_n(A).$$

Thus,

$$\int_E f\, d\mu = \mu(A) = \sum_{n=1}^{\infty} \mu_n(A) = \sum_{n=1}^{\infty} \int_E f\, d\mu_n.$$

Suppose now that $f = \sum_{k=1}^{m} a_k 1_{A_k}$ is a non-negative simple function, where $a_1, \cdots, a_m$ are distict and $A_1, \cdots, A_m$ are measurable and disjoint. Then,

$$\int_E f\, d\mu = \sum_{k=1}^{m} a_k \mu(A_k) = \sum_{k=1}^{m} a_k \sum_{n=1}^{\infty} \mu_n(A_k) = \sum_{n=1}^{\infty}\sum_{k=1}^{m} a_k \mu_n(A_k) = \sum_{n=1}^{\infty} \int_E f\, d\mu_n.$$

Finally, let $f \geq 0$ be measurable. There exists an increasing sequence of non-negative simple functions f_m converging to f. Then,

$$\int_E f\, d\mu = \lim_{m\to\infty} \int_E f_m\, d\mu = \lim_{m\to\infty} \sum_{n=1}^{\infty} \int_E f_m\, d\mu_n = \sum_{n=1}^{\infty} \lim_{m\to\infty} \int_E f_m\, d\mu_n = \sum_{n=1}^{\infty} \int_E f\, d\mu_n.$$

Notice that problem 2(b) of Exercises 10 allows us to interchange the limit and the summation since for each n, the sequence $(\int_E f_m\, d\mu_n)$ is increasing.

3. Let $(E_1, \mathcal{B}_1, \mu_1)$ and $(E_2, \mathcal{B}_2, \mu_2)$ be σ-finite measure spaces. Let $\Gamma \in \mathcal{B}_1 \times \mathcal{B}_2$. For $x_1 \in E_1$, $x_2 \in E_2$, let $\Gamma(x_1) = \{x_2 \in E_2 : (x_1, x_2) \in \Gamma\}$ and $\Gamma(x_2) = \{x_1 \in E_1 : (x_1, x_2) \in \Gamma\}$. Show that the following are equivalent:

   (i) $(\mu_1 \times \mu_2)(\Gamma) = 0$,

   (ii) $\mu_1(\Gamma(x_2)) = 0$ for μ_2 almost every $x_2 \in E_2$,

   (iii) $\mu_2(\Gamma(x_1)) = 0$ for μ_1 almost every $x_1 \in E_1$.

   Proof By Tonelli's Theorem.

   $$(\mu_1 \times \mu_2)(\Gamma) = \int_{E_2} \mu_1(\Gamma(x_2))\, d\mu_2(x_2) = \int_{E_1} \mu_2(\Gamma(x_1))\, d\mu_1(x_1).$$

   Since $\mu_1(\Gamma(x_2))$ and $\mu_2(\Gamma(x_1))$ are non-negative measurable functions on $(E_2, \mathcal{B}_2, \mu_2)$ and $(E_1, \mathcal{B}_1, \mu_1)$ respectively, it follows from Theorem 3.2.8 that $(\mu_1 \times \mu_2)(\Gamma) = 0$ if and only if $\mu_1(\Gamma(x_2)) = 0$ for μ_2 almost every $x_2 \in E_2$, and $(\mu_1 \times \mu_2)(\Gamma) = 0$ if and only if $\mu_2(\Gamma(x_1)) = 0$ for μ_1 almost every $x_1 \in E_1$.

4. Consider $(\mathbb{R}, \mathcal{B}, \lambda)$, where $\mathcal{B}$ is the Borel σ-algebra, λ is Lebesgue measure and μ is counting measure (i.e. $\mu(A) =$ number of elements in A). Let $A = \{x, y) : x = y\}$, show that

   $$\int_{\mathbb{R}} \int_{\mathbb{R}} 1_A(x,y) d\lambda(x) d\mu(y) = 0$$

while
$$\int_{\mathbb{R}} \int_{\mathbb{R}} 1_A(x, y) d\mu(y) d\lambda(x) = \infty.$$
Why does not this violate Tonelli's Theorem?

Proof For any $x \in \mathbb{R}$, $A(x) = \{y \in \mathbb{R} : (x, y) \in A\} = \{x\}$. Thus, $\mu(A(x)) = 1$ and $\lambda(A(x)) = 0$. Hence,
$$\int_{\mathbb{R}} \int_{\mathbb{R}} 1_A(x, y) d\lambda(x) d\mu(y) = \int_{\mathbb{R}} \lambda(A(y)) d\mu(y) = 0,$$
and
$$\int_{\mathbb{R}} \int_{\mathbb{R}} 1_A(x, y) d\mu(y) d\lambda(x) = \int_{\mathbb{R}} \mu(A(x)) d\lambda(x) = \lambda(\mathbb{R}) = \infty.$$
The reason why Tonelli's Theorem does not hold is because the measure μ is **not** σ-finite.