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Measure and Integration Solutions 14

1. $(E, \mathcal{B}, \mu)$ be a σ-finite measure space, and $f : E \to [0, \infty)$ measurable. Define

$$\Gamma(f) = \{(x, t) \in E \times [0, \infty) : t < f(x)\},$$

and

$$\overline{\Gamma}(f) = \{(x, t) \in E \times [0, \infty) : t \leq f(x)\}.$$

(a) Show that the function $F : E \times [0, \infty) \to \mathbb{R}$ given by $F(x, t) = f(x) - t$ is measurable with respect to the product σ-algebra $\mathcal{B} \times \mathcal{B}_{[0,\infty)}$, where $\mathcal{B}_{[0,\infty)}$ is the restriction of the Borel σ-algebra on $[0, \infty)$.

(b) Show that $\Gamma(f), \overline{\Gamma}(f) \in \mathcal{B} \times \mathcal{B}_{[0,\infty)}$, and

$$(\mu \times \lambda_{\mathbb{R}})(\Gamma(f)) = (\mu \times \lambda_{\mathbb{R}})(\overline{\Gamma}(f)) = \int_E f(x)\, d\mu(x).$$

Proof (a) We will show that the function F is the composition of measurable functions. Let $f_1, f_2 : E \times [0, \infty) \to [0, \infty)$ be given by

$$f_1(x, t) = f(x), \text{ and } f_2(x, t) = t.$$

Then, for any $a \geq 0$,

$$f_1^{-1}([a, \infty)) = f^{-1}([a, \infty)) \times \mathbb{R} \in \mathcal{B} \times \mathcal{B}_{[0,\infty)}, \text{ and } f_2^{-1}([a, \infty)) = E \times [a, \infty) \in \mathcal{B} \times \mathcal{B}_{[0,\infty)}.$$

Thus, f_1, f_2 are measurable. By Lemma 3.2.2, the tensor product $(f_1 \times f_2) : E \times [0, \infty) \to [0, \infty) \times [0, \infty)$ given by $(f_1 \times f_2)(x, t) = (f_1(x, t), f_2(x, t)) = (f(x), t)$ is measurable. Let $g : [0, \infty) \times [0, \infty) \to \mathbb{R}$ be given by $g(s, t) = s - t$, then g is continuous, and hence measurable. Now, $F(x, t) = g \circ (f_1 \times f_2)(x, t)$, hence F is the composition of two measurable functions, therefore F is measurable.

Proof (b) Notice that $\Gamma(f) = F^{-1}((0, \infty))$ and $\overline{\Gamma}(f) = F^{-1}([0, \infty))$. Since F is measurable, it follows that $\Gamma(f), \overline{\Gamma}(f) \in \mathcal{B} \times \mathcal{B}_{[0,\infty)}$.

Since $1_{\Gamma(f)}, 1_{\overline{\Gamma}(f)} \geq 0$ are measurable, by Tonelli's Theorem (Theorem 4.1.5),

$$\begin{aligned}
(\mu \times \lambda_{\mathbb{R}})(\Gamma(f)) &= \int_{E \times [0,\infty)} 1_{\Gamma(f)}(x, t) d(\mu \times \lambda_{\mathbb{R}})(x, t) \\
&= \int_E \int_{[0,\infty)} 1_{\{t \geq 0:\, t < f(x)\}}(t) d\lambda_{\mathbb{R}}(t) d\mu(x) \\
&= \int_E \lambda_{\mathbb{R}}([0, f(x)))\, d\mu(x) \\
&= \int_E f(x) d\mu(x).
\end{aligned}$$

Similarly,

$$(\mu \times \lambda_{\mathbb{R}})(\overline{\Gamma}(f)) = \int_E \lambda_{\mathbb{R}}\left([0, f(x)]\right) d\mu(x) = \int_E f(x) d\mu(x).$$

2. Let $E = \{(x, y) : 0 < x < \infty, 0 < y < 1\}$. We consider on E the restriction of the product Borel σ-algebra, and the restriction of the product Lebesgue measure $\lambda \times \lambda$. Let $f : E \to \mathbb{R}$ be given by $f(x, y) = y \sin x \, e^{-xy}$.

   (a) Show that f is $\lambda \times \lambda$ integrable on E.

   (b) Applying Fubini's Theorem to the function f, show that

$$\int_0^\infty \frac{\sin x}{x} \left(\frac{1 - e^{-x}}{x} - e^{-x} \right) dx = \frac{1}{2} \log 2.$$

Proof(a) Notice that f is continuous, and hence measurable. Furthermore, $|f(x, y)| \leq ye^{-xy}$. The fuction $g(x, y) = ye^{-xy}$ is non-negative measurable function, hence by Tonelli's Theorem,

$$\begin{aligned} \int_E |f(x, y)| d(\lambda \times \lambda)(x, y) &\leq \int_E ye^{-xy} d(\lambda \times \lambda)(x, y) \\ &= \int_0^1 \int_0^\infty ye^{-xy} dx dy \\ &= \int_0^1 1 \, dy = 1. \end{aligned}$$

Notice that the integrands are Riemann integrable, hence the Riemann integral equals the Lebesgue integral, also the second equality is obtained by integration by parts. This shows that f is $\lambda \times \lambda$ integrable on E.

Proof(b) By Fubini's Theorem,

$$\int_E f(x, y) d(\lambda \times \lambda)(x, y) = \int_0^1 \int_0^\infty y \sin x \, e^{-xy} dx dy = \int_0^\infty \int_0^1 y \sin x \, e^{-xy} dy dx.$$

Using integration by parts, one has

$$\int_0^\infty y \sin x \, e^{-xy} dx = \frac{y}{y^2 + 1}.$$

Hence,

$$\int_E f(x, y) d(\lambda \times \lambda)(x, y) = \int_0^\infty \frac{y}{y^2 + 1} dy = \frac{1}{2} \log 2.$$

On the other hand, again by integration by parts one has,

$$\int_0^1 y \sin x \, e^{-xy} dy = \frac{\sin x}{x} \left(\frac{1 - e^{-x}}{x} - e^{-x} \right).$$

Therefore,

$$\int_0^\infty \frac{\sin x}{x} \left(\frac{1 - e^{-x}}{x} - e^{-x} \right) dx = \frac{1}{2} \log 2.$$

3. Let $(L, (\ ,\ ))$ be an inner product space, and let $||x||_L = (x,x)^{1/2}$. $x \in L$.

   (a) Let $(x_n) \subseteq L$, and $x \in L$. Show that if $\lim_{n\to\infty} ||x_n - x||_L = 0$, then $\lim_{n\to\infty} ||x_n||_L = ||x||_L$.

   (b) Prove that the inner product $(\ ,\ )$ is jointly continuous, i.e. if $\lim_{n\to\infty} ||x_n - x||_L = 0$ and $\lim_{n\to\infty} ||y_n - y||_L = 0$, then $\lim_{n\to\infty} (x_n, y_n) = (x, y)$.

Proof (a)

$$|\, ||x_n||_L - ||x||_L \,| \leq ||x_n - x||_L \to 0 \text{ as } n \to \infty.$$

Thus, $\lim_{n\to\infty} ||x_n||_L = ||x||_L$.

Proof (b) Suppose that $\lim_{n\to\infty} ||x_n - x||_L = 0$ and $\lim_{n\to\infty} ||y_n - y||_L = 0$. By Cauchy-Schwartz inequality and part (a), we have

$$\begin{aligned}
|(x_n, y_n) - (x, y)| &= |(x_n, y_n - y) + (x_n - x, y)| \\
&\leq |(x_n, y_n - y)| + |(x_n - x, y)| \\
&\leq ||x_n||_L ||y_n - y||_L + ||y||_L ||x_n - x||_L \\
&\to ||x_n||_L \cdot 0 + ||y||_L \cdot 0 = 0.
\end{aligned}$$

Therefore, $\lim_{n\to\infty} (x_n, y_n) = (x, y)$.

4. Let $(E, \mathcal{B}, \mu)$ be a measure space, and let $\{f_n\} \subseteq L^2(\mu)$ be such that

$$\lim_{m\to\infty} \sup_{n\geq m} ||f_n - f_m||_{L^2(\mu)} = 0.$$

Show that there exists a function $f \in L^2(\mu)$ such that $\lim_{n\to\infty} ||f_n - f||_{L^2(\mu)} = 0$. In other words $(L^2(\mu), ||\ ||_{L^2(\mu)})$ is a complete metric space.

Proof By the Markov inequality,

$$\mu(|f_n - f_m| \geq \epsilon) = \mu(|f_n - f_m|^2 \geq \epsilon^2) \leq \frac{1}{\epsilon^2} ||f_n - f_m||^2_{L^2(\mu)}.$$

Hence,

$$\lim_{m\to\infty} \sup_{n\geq m} \mu(|f_n - f_m| \geq \epsilon) \leq \lim_{m\to\infty} \sup_{n\geq m} \frac{1}{\epsilon^2} ||f_n - f_m||^2_{L^2(\mu)} = 0.$$

By Theorem 3.3.10 there exists a measurable function f such that $f_n \to f$ in μ-measure. Furthermore, there exists a subsequence (f_{n_i}) such that $f_{n_i} \to f$ μ a.e., hence for each m, $f_{n_i} - f_m \to f - f_m$ (as $n \to \infty$) μ a.e.. By Fatou's lemma

$$||f - f_m||^2_{L^2(\mu)} \leq \liminf_{i\to\infty} ||f_{n_i} - f_m||^2_{L^2(\mu)} \leq \sup_{n\geq m} ||f_n - f_m||^2_{L^2(\mu)}.$$

Thus, $\lim_{m\to\infty} ||f - f_m||_{L^2(\mu)} = 0$. Furthermore, $f - f_m \in L^2(\mu)$ for each m. Since $f = (f - f_m) + f_m$ with $f - f_m \in L^2(\mu)$ and $f_m \in L^2(\mu)$, it follows that $f \in L^2(\mu)$.