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# Measure and Integration Solutions 15

1. Let $(E, \mathcal{B}, \mu)$ be a measure space. Show (without using Cauchy-Schwartz inequality) that if $f, g \in L^2(\mu)$, then

$$\int_E |fg|\, d\mu \leq ||f||_{L^2(\mu)}\, ||g||_{L^2(\mu)}.$$

This is known as Hölders inequality. (Hint: for any real numbers $a, b$ one has $2|ab| \leq a^2 + b^2$, why?)

**Proof** Let $a = \dfrac{|f|}{||f||_{L^2(\mu)}}$ and $b = \dfrac{|g|}{||g||_{L^2(\mu)}}$. Using the hint, one has

$$\frac{|fg|}{||f||_{L^2(\mu)}||g||_{L^2(\mu)}} \leq \frac{1}{2}\frac{|f|^2}{||f||^2_{L^2(\mu)}} + \frac{|g|^2}{||g||^2_{L^2(\mu)}}.$$

Integrating both sides ( the integral of the right hand side is equal to one) and multiplying by $||f||_{L^2(\mu)}||g||_{L^2(\mu)}$, one gets the required result.

2. Let $(E, \mathcal{B}, \mu)$ be a finite measure space. Show that $L^2(\mu) \subseteq L^1(\mu)$. Show that the result is not true in case $\mu$ is not a finite measure

**Proof** Let $f \in L^2(\mu)$, then by Hölders inequality,

$$\int_E |f|\, d\mu = \int_E |f| \cdot 1\, d\mu \leq ||f||_{L^2(\mu)}||1||_{L^2(\mu)} = ||f||_{L^2(\mu)}\mu(E) < \infty.$$

Thus, $f \in ||f||_{L^1(\mu)}$, and hence $L^2(\mu) \subseteq L^1(\mu)$.

We give a simple counterexample to show that the result is not true if $\mu$ is not a finite measure. For this consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$, where $\lambda$ is Lebesgue measure. Let $f = \dfrac{1}{x} \cdot 1_{[1,\infty)}$. Then $\int_{\mathbb{R}} f d\lambda = \infty$, while $\int_{\mathbb{R}} f^2 d\lambda = 1$. This shows that $f \in L^2(\mu)$ but $f \notin L^1(\mu)$

3. Let $\mu$ and $\nu$ be two measures on the measure space $(E, \mathcal{B})$ such that $\mu(A) \leq \nu(A)$ for all $A \in \mathcal{B}$. Show that if $f$ is any non-negative measurable function on $(E, \mathcal{B})$, then $\int_E f\, d\mu \leq \int_E f\, d\nu$. Conclude that if $\nu$ is a finite measure, then $L^2(\nu) \subseteq L^1(\nu) \subseteq L^1(\mu)$.

**Proof** Suppose first that $f = 1_A$ is the indicator function of some set $A \in \mathcal{B}$. Then

$$\int_E f\, d\mu = \mu(A) \leq \nu(A) = \int_E f\, d\nu.$$

Suppose now that $f = \sum_{k=1}^{n} \alpha_k 1_{A_k}$ is a non-negative measurable step function. Then,

$$\int_E f \, d\mu = \sum_{k=1}^{n} \alpha_k \mu(A_k) \leq \sum_{k=1}^{n} \alpha_k \nu(A_k) = \int_E f \, d\nu.$$

Finally, let $f$ be a non-negative measurable function, then there exists a sequence of non-negative measurable step functions $f_n$ such that $f_n \uparrow f$. By the Monotone Convergence Theorem,

$$\int_E f \, d\mu = \lim_{n\to\infty} \int_E f_n \, d\mu \leq \lim_{n\to\infty} \int_E f_n \, d\nu = \int_E f \, d\nu.$$

The above implies that if $f \in L^1(\nu)$, then $f \in L^1(\mu)$, i.e. $L^1(\nu) \subseteq L^1(\mu)$.

If $\nu$ is a finite measure, then by problem 3 and the above, we have $L^2(\nu) \subseteq L^1(\nu) \subseteq L^1(\mu)$.

4. Let $(E, \mathcal{B})$ be a measurable space, and $\mu_1$, $\mu_2$ and $\nu$ measures on $(E, \mathcal{B})$. Show the following:

   (a) If $\mu_1 \perp \nu$ and $\mu_2 \perp \nu$, then $\mu_1 + \mu_2 \perp \nu$.

   (b) If $\mu_1 \ll \nu$ and $\mu_2 \perp \nu$, then $\mu_1 \perp \mu_2$.

   (c) If $\mu_1 \ll \nu$ and $\mu_1 \perp \nu$, then $\mu_1$ is the zero measure.

**Proof (a)** Let $A, B \in \mathcal{B}$ be such that

$$\nu(A^c) = \mu_1(A) = 0 \text{ and } \nu(B^c) = \mu_2(B) = 0.$$

Let $C = A \cap B$, then

$$\nu(C^c) \leq \nu(A^c) + \nu(B^c) = 0$$

and

$$(\mu_1 + \mu_2)(C) \leq \mu_1(A) + \mu_2(B) = 0.$$

Thus, $\nu(C^c) = 0 = (\mu_1 + \mu_2)(C)$ and $\mu_1 + \mu_2 \perp \nu$.

**Proof (b)** Let $A \in \mathcal{B}$ be such that $\nu(A^c) = 0 = \mu_2(A)$. Since $\mu_1 \ll \nu$, then $\mu_1(A^c) = 0 = \mu_2(A)$. Thus, $\mu_1 \perp \mu_2$.

**Proof (c)** Let $A \in \mathcal{B}$ be such that $\nu(A^c) = 0 = \mu_1(A)$. Since $\mu_1 \ll \nu$, it follows that $\mu_1(A^c) = 0 = \mu_1(A)$. Then, $\mu_1(E) = \mu_1(A) + \mu_1(A^c) = 0$. Thus $\mu_1(B) = 0$ for all $B \in \mathcal{B}$.